VOLUME II
FLYING QUALITIES PHASE

# CHAPTER 2
# VECTORS AND MATRICES



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## INTRODUCTION

This chapter studies the algebra and calculus of vectors and matrices, as specifically applied to the USAF Test Pilot School curriculum. The course is a prerequisite for courses in Equations of Motion, Dynamics, Linear Control Systems, Flight Control Systems, and Inertial Navigation Systems. The course deals only with applied mathematics; therefore, the theoretical scope of the subject is limited.

The text begins with the definition of determinants as a prerequisite to the remainder of the text. Vector analysis follows with rigid body kinematics introduced as an application. The last section deals with matrices.

## 2.2 DETERMINANTS

A determinant is a function which associates a number (real, imaginary, or vector) with every square array (n columns and n rows) of numbers. The determinant is denoted by vertical bars on either side of the array of numbers. Thus, if A is an (n x n) array of numbers where i designates rows and j designates columns, the determinant of A is written

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

### 2.2.1 First Minors and Cofactors

When the elements of the $i^{th}$ row and $j^{th}$ column are removed from a (n x n) square array, the determinant of the remaining (n-1) x (n-1) square array is called a first minor of A and is denoted by $M_{ij}$. It is also called the minor of $a_{ij}$. The signed minor, with the sign determined by the sum of

the row and column, is called the cofactor of $a_{ij}$ and is denoted by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Example:

$$\text{If } |A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then,

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Also, $A_{11} = (-1)^{1+1} M_{11} = (+1) M_{11}$ and $A_{32} = (-1)^{3+2} M_{32} = (-1) M_{32}$

2.2.2 Determinant Expansion

The determinant is equal to the sum of the products of the elements of any single row or column and their respective cofactors; i.e.,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}, \text{ for any single } i^{th} \text{ row.}$$

or

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}, \text{ for any single } j^{th}$$

column.

2.2.2.1 Expanding a 2 x 2 Determinant. Expanding a 2 x 2 determinant about the first row is the easiest. The sign of the cofactor of an element can be determined quickly by observing that the sums of the subscripts alternate from even to odd when advancing across rows or down columns, meaning the signs

alternate also. For example,

$$\text{if} \quad |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

the signs of the associated cofactors alternate as shown,

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

By deleting the row and column of $a_{11}$, we find its cofactor is just the element $a_{22}$ [actually $(+1) \times a_{22}$] for a 2 x 2 array, and likewise the cofactor for $a_{12}$ is $(-a_{21})$ [or $(-1) \times a_{21}$]. The sum of the two products of the two diagonals gives us the expansion or value of the determinant.

$$|A| = a_{11}\ A_{11} + a_{12}\ A_{12} = a_{11}\ a_{22} + a_{12}\ (-a_{21}) = a_{11}\ a_{22} - a_{12}\ a_{21}$$

This simple example has been shown for clarity. Actual calculation of a 2 x 2 determinant is easy if we just remember it as the subtraction of the cross multiplication of the elements. For example,

$$|R| = \begin{array}{c} \begin{array}{cc} (+) & (-) \end{array} \\ \begin{vmatrix} 8 & 3 \\ 6 & 5 \end{vmatrix} \end{array} = (8)(5)-(3)(6) = 22$$

2.2.2.2 <u>Expanding a 3 x 3 Determinant</u>.

$$|A| = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding $|A|$ about the first row gives

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} =$$

$$a_{11} \; (+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \; (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \; (+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11} (a_{22} a_{33} - a_{23} a_{32}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (a_{21} a_{32} - a_{22} a_{31})$$

Expanding and grouping like signs,

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$
$$-a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

Close inspection of the last equation shows another method for 3 x 3 determinants using diagonal multiplication. If the first two columns are appended to the determinant, six sets of diagonals are used to find the six terms above. The signs are determined by the direction of the diagonal as shown in the illustration.

$$A = \begin{array}{|ccc|cc} (+) & (+) & (+) & & \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ (-) & (-) & (-) & & \end{array}$$

For example,

$$A = \begin{array}{|ccc|cc} (+) & (+) & (+) & & \\ 2 & -1 & 6 & 2 & -1 \\ 3 & 5 & 4 & 3 & 5 \\ 1 & -2 & 3 & 1 & -2 \\ (-) & (-) & (-) & & \end{array}$$

$$= (2)(5)(3) + (-1)(4)(1) + (6)(3)(-2) - (6)(5)(1) - (2)(4)(-2) - (-1)(3)(3)$$

$$= 30 + (-4) + (-36) - 30 - (-16) - (-9) = -40 + 25 = \underline{-15}$$

The quicker methods of calculating determinants are useful for the two simple cases here. The row expansion method will be more useful for calculating vector cross products. The use of determinants for solving sets of linear equations will be discussed later in this chapter in the matrix section. Determinants will also be used in solving sets of linear differential equations in Chapter 3, Differential Equations.

While the general tool for evaluating determinants by hand calculation is simple, for determinants of greater size the calculations are lengthy. A 5 x 5 determinant would contain 120 terms of 5 factors each. Evaluating larger determinants is an ideal task for the computer, and standard programs are available for this task.

## 2.3 VECTOR AND SCALAR DEFINITIONS

In general, a vector can be defined as an ordered set of "n" quantities such as $\langle a_1, a_2, a_3, \ldots, a_n \rangle$. In TPS, vector analysis will be limited to two-and three-dimensional space. Thus, $x\bar{i} + y\bar{j}$ and $x\bar{i} + y\bar{j} + z\bar{k}$ are representations of vectors in each space, while $x\bar{i}$, $y\bar{j}$, and $z\bar{k}$ are referred to as components of the vector.

Physically, a vector is an entity such as force, velocity, or acceleration, which possesses both magnitude and direction. This is the usual approach in applied physics and engineering, and the results can be directly applied to courses here at the School.

Almost any physical discussion will involve, in addition to vectors, entities such as volume, mass, and work, which possess only magnitude and are known as scalars. To distinguish vectors from scalars, a vector quantity will be indicated by putting a line above the symbol; thus $\bar{F}$, $\bar{v}$, and $\bar{a}$ will be used to represent force, velocity, and acceleration, respectively.

The magnitude of vector $\bar{F}$ is indicated by enclosing the symbol for the vector between absolute value bars, $|\bar{F}|$. Graphically, a scalar quantity can be adequately represented by a mark on a fixed scale. To represent a vector quantity requires a directed line segment whose direction is the same as the direction of the vector and whose measured length is equal to the magnitude of the vector.

The direction of a vector is determined by a single angle in two dimensions and two angles in three dimensions, angles whose cosines are called direction cosines. This text will not deal directly with direction cosines, so no example is necessary.

### 2.3.1 Vector Equality

Two vectors whose magnitude and direction are equal are said to be equal. If two vectors have the same length but the opposite direction, either is the negative of the other. This is true even when graphically two vectors are not physically drawn from the same starting point.

A vector that may be drawn from any starting point is called a free vector. However, when applied in a problem, the position of a vector may be important. For instance, in Figure 2.1, the distance of the line of application of a force from the center of gravity of a rigid body is critical if calculating moments, although the actual point of application along the line isn't critical.

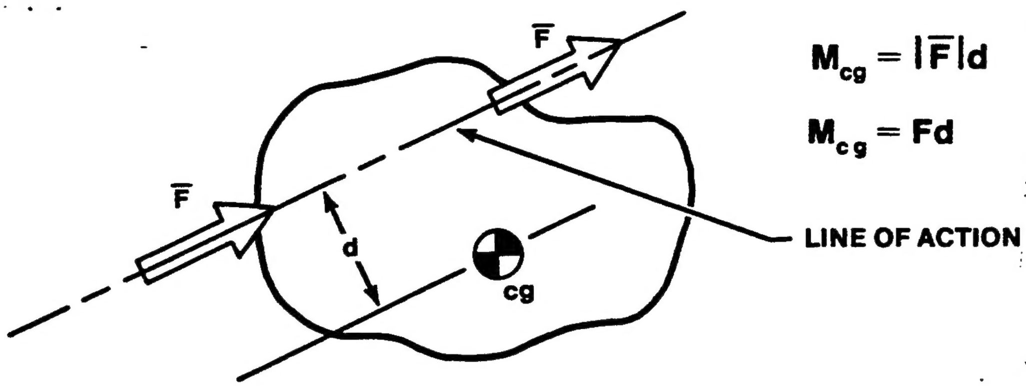



FIGURE 2.1. MOMENT CALCULATION

For other applications, the point of action as well as the line of action must be fixed. Such a vector is usually referred to as bound. The velocity of the satellite in the orbital mechanics problem shown in Figure 2.2 is an example of a bound vector.

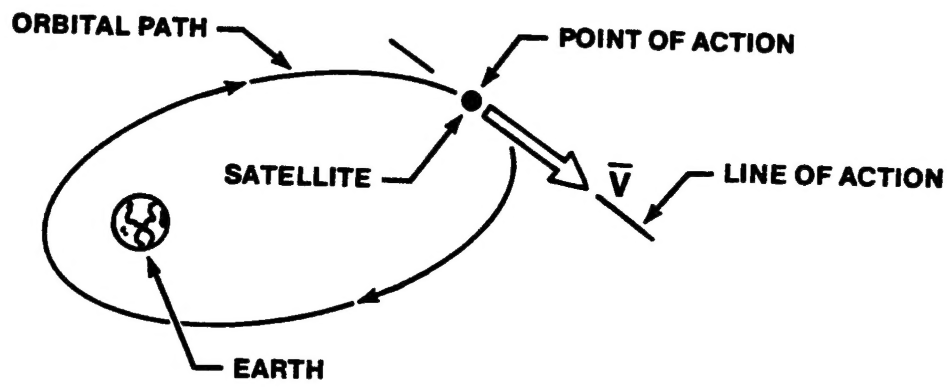



FIGURE 2.2. EXAMPLE OF A BOUND VECTOR

### 2.3.2 Vector Addition

Graphically, the sum of two vectors $\overline{A}$ and $\overline{B}$ is defined by the familiar parallelogram law; i.e., if $\overline{A}$ and $\overline{B}$ are drawn from the same point or origin, and if the parallelogram having $\overline{A}$ and $\overline{B}$ as adjacent sides is constructed, then the sum $\overline{A} + \overline{B}$ can be defined as the vector represented by the diagonal of this parallelogram which passes through the common origin of $\overline{A}$ and $\overline{B}$. Vectors can also be added by drawing them "nose-to-tail." See Figure 2.3.

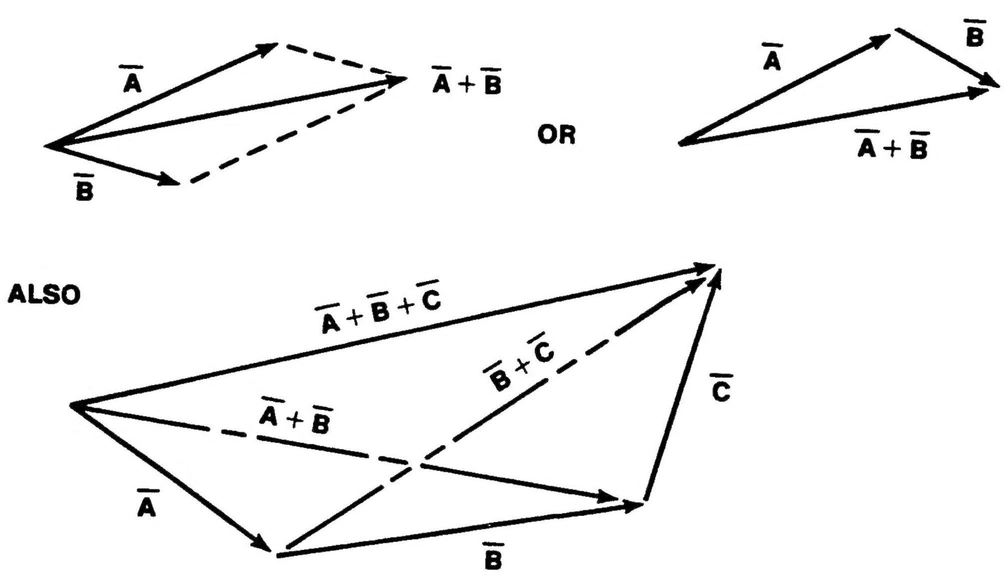



FIGURE 2.3. ADDITION OF VECTORS

Graphically from Figure 2.3, it is evident that vector addition is commutative and associative, respectively,

$$\bar{A} + \bar{B} = \bar{B} + \bar{A} \quad \text{and} \quad \bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$$

### 2.3.3 Vector Subtraction

Vector subtraction is defined as the difference of two vectors A and B,

$$\bar{A} - \bar{B} = \bar{A} + (-\bar{B}),$$

where

$$(-1)(\bar{B}) = (-\bar{B})$$

and is defined as a vector with the same magnitude but opposite direction. See Figure 2.4. This introduces the necessity for vector-scalar multiplication.

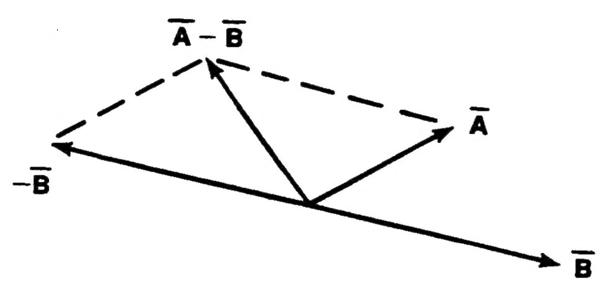



FIGURE 2.4. VECTOR SUBTRACTION

### 2.3.4 Vector-Scalar Multiplication

The product of a vector and a scalar follows algebraic rules. The product of a scalar m and a vector $\bar{A}$ is the vector $m\bar{A}$, whose length is the product of the absolute value of m and magnitude of $\bar{A}$, and whose direction is the same as the direction of $\bar{A}$, if m is positive, and opposite to it, if m is negative.

### 2.3.5 Unit and Zero Vectors

Regardless of its direction, a vector whose length is one (unity) is called a unit vector. If $\bar{a}$ is a vector with magnitude other than zero, then unit vector $\hat{a}$ is defined as $\bar{a}/|\bar{a}|$, where $\hat{a}$ is a unit vector having the same direction as $\bar{a}$ and magnitude of one. It happens that the components of a unit vector are also the cosines of the angles necessary to define the direction. Unit vectors in the body axis coordinate system will retain the bar symbol; i.e., $\bar{i}$, $\bar{j}$ and $\bar{k}$.

The zero vector has zero magnitude and in this text has any direction. It is notationally correct to designate the zero vector with a bar, $\bar{0}$.

## 2.4 LAWS OF VECTOR - SCALAR ALGEBRA

If $\bar{A}$, $\bar{B}$, and $\bar{C}$ are vectors and m and n are scalars, then

| | | |
|---|---|---|
| 1. | $m\bar{A} = \bar{A}m$ | Commutative Multiplication |
| 2. | $m(n\bar{A}) = (mn)\bar{A}$ | Associative Multiplication |
| 3. | $(m + n)\bar{A} = m\bar{A} + n\bar{A}$ | Distributive |
| 4. | $m(\bar{A} + \bar{B}) = m\bar{A} + m\bar{B}$ | Distributive |

These laws involve multiplication of a vector by one or more scalars. Products of vectors will be defined later.

These laws, along with the vector addition laws already introduced, enable vector equations to be treated the same way as ordinary scalar algebraic equations. For example,

$$\text{if } \bar{A} + \bar{B} = \bar{C}$$

then by algebra

$$\bar{A} = \bar{C} - \bar{B}$$

### 2.4.1 Vectors in Coordinate Systems

The right-handed rectangular coordinate system is used unless otherwise stated. Such a system derives its name from the fact that a right threaded screw rotated through 90° in the direction from the positive x-axis to the positive y-axis will advance in the positive z direction, as shown in Figure 2.5. Practically, curl the fingers of the right hand in a direction from the positive x-axis to the positive y-axis, and the thumb will point in the positive z-axis direction.

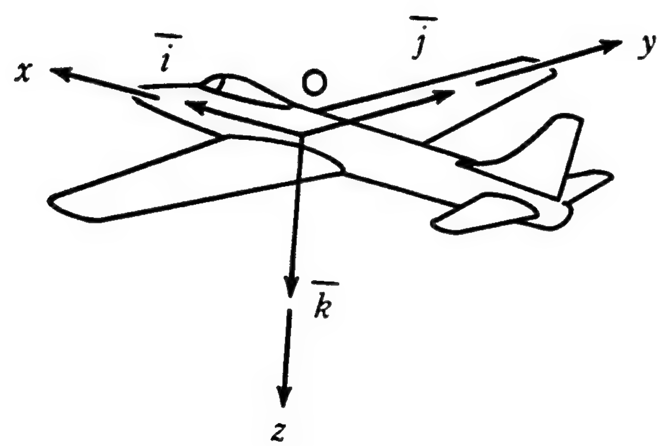



FIGURE 2.5. RIGHT-HANDED COORDINATE SYSTEM

An important set of unit vectors are those having the directions of the positive x, y, and z axes of a three-dimensional rectangular coordinate system and are denoted $\bar{i}$, $\bar{j}$, and $\bar{k}$, respectively, as shown in Figure 2.5.

Any vector in three dimensions can be represented with initial point at the origin of a rectangular coordinate system as shown in Figure 2.6. The perpendicular projection of the vector on the axes gives the vector's components on the axes. Multiplying the scalar magnitude of the projection by the appropriate unit vector in the direction of the axis gives a component vector of the original vector. Note that summing the component vectors graphically gives the original vector as a resultant.

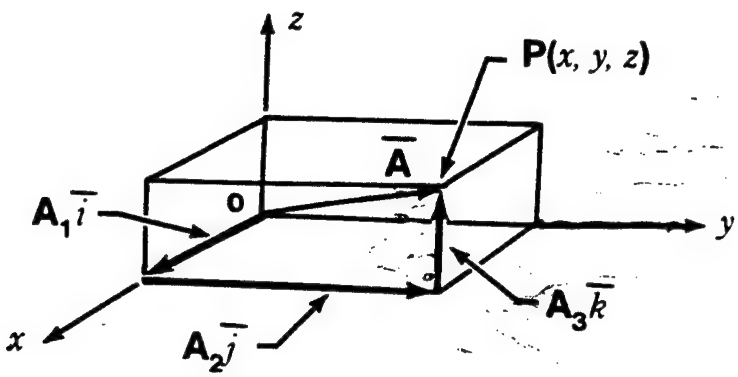



FIGURE 2.6. COMPONENTS OF A VECTOR

In Figure 2.6 the component vectors are $A_1\bar{i}$, $A_2\bar{j}$, and $A_3\bar{k}$. The sum or resultant of the components gives a new notation for a vector in terms of its components.

$$\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$$

After noticing that the coordinates of the end-point of a vector $\bar{A}$ whose tail is at the origin are equal to the components of the vector itself ($A_1 = x$, $A_2 = y$, and $A_3 = z$), the vector may be more easily written as

$$\bar{A} = x\bar{i} + y\bar{j} + z\bar{k}$$

The vector from the origin to a point in a coordinate system is called a <u>position vector</u>, so the vector notation above is also the position vector for the point P. The same definitions for notation, components, and position hold for a two-dimensional system with the third component eliminated.

The magnitude is easily calculated as,

$$A = \sqrt{A_1^{\,2} + A_2^{\,2} + A_3^{\,2}} \qquad \text{or} \qquad A = \sqrt{x^2 + y^2 + z^2}$$

An arbitrary vector from initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ such as shown in Figure 2.7 can be represented in terms of unit vectors, also.

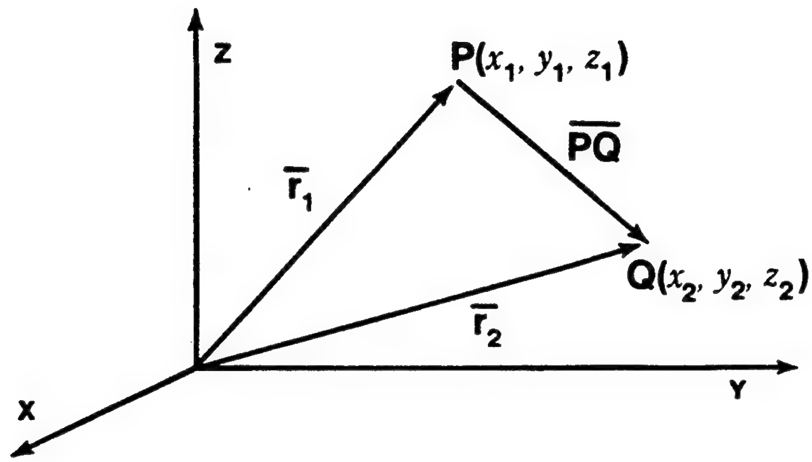



FIGURE 2.7. ARBITRARY VECTOR REPRESENTATION

First write the position vectors for the two points P and Q.

$$\overline{r}_1 = x_1\overline{i} + y_1\overline{j} + z_1\overline{k}$$

and

$$\overline{r}_2 = x_2\overline{i} + y_2\overline{j} + z_2\overline{k}$$

Then using addition,

$$\overline{r}_1 + \overline{PQ} = \overline{r}_2$$

or

$$\overline{PQ} = \overline{r}_2 - \overline{r}_1 = (x_2 - x_1)\overline{i} + (y_2 - y_1)\overline{j} + (z_2 - z_1)\overline{k}$$

### 2.4.2 Dot Product

In addition to the product of a scalar and a vector, two other types of products are defined in vector analysis. The first of these is the dot, or scalar product, denoted by a dot between the two vectors. The dot product is an operation between two vectors, and results in a scalar (thus the name scalar product). Analytically, it is calculated by adding the products of like components. This is, if

$$\bar{A} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

and

$$\bar{B} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$$

then

$$\bar{A}\cdot\bar{B} = a_1b_1 + a_2b_2 + a_3b_3$$

which is a real number or scalar.

Geometrically, it is equal to the product of the magnitudes of two vectors and the cosine of the angle between them (the angle is measured in the plane formed by the two vectors, if they had the same origin). The dot product is written

$$\bar{A}\cdot\bar{B} = |\bar{A}|\ |\bar{B}|\cos\Theta$$

Example

$$(2\bar{i} - 1\bar{j} + 4\bar{k})\cdot(-\bar{i} + 3\bar{j} + 5\bar{k}) = (2)(-1) + (-1)(3) + (4)(5) = 15$$

The magnitudes are

$$\sqrt{4 + 1 + 16} = 4.6$$

and

$$\sqrt{1 + 9 + 25} = 5.9$$

Therefore, solving for

$$\cos\Theta = 15 / (4.6)(5.9) = 15/27.1 = 0.553$$

so

$$\Theta = 56.1°$$

Some interesting applications of the dot product are the geometric implications. For instance, the geometric, scalar projection of one vector on another is shown on Figure 2.8.

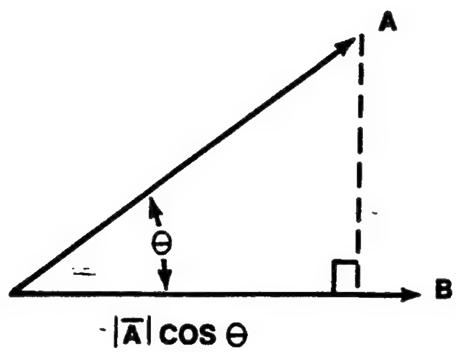



FIGURE 2.8. GEOMETRIC PROJECTION OF VECTORS

Using trigonometry, the projection of $\bar{A}$ on $\bar{B}$ is seen to be equal to $|\bar{A}| \cos \Theta$. A quick method to calculate such a projection without knowing the angle is to calculate the dot product and divide by the magnitude of the vector projected on to. That is, the projection of $\bar{A}$ on to $\bar{B}$ is equal to $\bar{A} \cdot \bar{B}/|\bar{B}| = |\bar{A}| \cos \Theta$.

Several particular dot products are worth mentioning. If one of the vectors is a unit vector, the dot product becomes

$$\bar{i} \cdot \bar{B} = |\bar{i}|\,|\bar{B}| \cos \Theta = (1)\,|\bar{B}| \cos \Theta = |\bar{B}| \cos \Theta,$$

which is the projection of $\bar{B}$ on $\bar{i}$ or more importantly the component of $\bar{B}$ in the direction of $\bar{i}$. Also note the dot product of a vector with itself is just equal to the magnitude squared, since the angle is zero and $\cos \Theta = 1$. More useful is the situation where two non-zero vectors are perpendicular (orthogonal). The dot product is zero because the cosine of 90 degrees is zero. Thus, for non-zero vectors the dot product may be a test of orthogonality. Examples of these properties using standard unit vectors are

$$\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$$

and

$$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0$$

### 2.4.3 Dot Product Laws

If $\overline{A}$, $\overline{B}$, and $\overline{C}$ are vectors and m is scalar, then

1. $\overline{A} \bullet \overline{B} = \overline{B} \bullet \overline{A}$ — Commutative Product
2. $\overline{A} \bullet (\overline{B} + \overline{C}) = \overline{A} \bullet \overline{B} + \overline{A} \bullet \overline{C}$ — Distributive Product
3. $m(\overline{A} \bullet \overline{B}) = (m\overline{A}) \bullet \overline{B} = \overline{A} \bullet (m\overline{B})$ — Associative Product

### 2.4.4 Cross Product

The third type of product involving vector operations is the cross, or vector product, denoted by placing an "X" between two vectors. By definition the cross product is an operation between two vectors which results in another vector (thus, vector product). Again both analytic and geometric definitions are given.

Analytically, the cross product is calculated for three-dimensional vectors by a top row expansion of a determinant.

$$\overline{A} \times \overline{B} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \overline{i} + (-1) \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \overline{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \overline{k}$$

$$= (a_2 b_3 - a_3 b_2)\overline{i} + (a_3 b_1 - a_1 b_3)\overline{j} + (a_1 b_2 - a_2 b_1)\overline{k}$$

For example,

$$(2\overline{i} + 4\overline{j} + 5\overline{k}) \times (3\overline{i} + \overline{j} + 6\overline{k}) =$$

$$\begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 2 & 4 & 5 \\ 3 & 1 & 6 \end{vmatrix}$$

$$= [(4)(6)-(5)(1)]\overline{i} - [(2)(6)-(3)(5)]\overline{j} + [(2)(1)-(3)(4)]\overline{k}$$

$$= \underline{19\overline{i} + 3\overline{j} - 10\overline{k}}$$

The geometrical definition has to be approached carefully because it must be remembered that the geometrical definition is <u>not</u> a vector. The magnitude (a scalar) of the cross product is equal to the product of the two magnitudes and the sine of the angle between the two vectors. Thus

$$\left|\bar{A} \times \bar{B}\right| = \left|\bar{A}\right| \left|\bar{B}\right| \sin \Theta$$

While the magnitude is determined as above, the direction of the resultant cross product vector is always orthogonal to the plane of the crossed vectors. The sense is such that when the fingers of the right hand are curled from the first vector to the second, through the smaller of the angles between the vectors, the thumb points in the direction of the cross product as shown in Figure 2.9. Note the importance in the order of writing $\bar{A} \times \bar{B}$ since $\bar{A} \times \bar{B} \neq \bar{B} \times \bar{A}$. That $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ is easily seen using the right-hand rule.

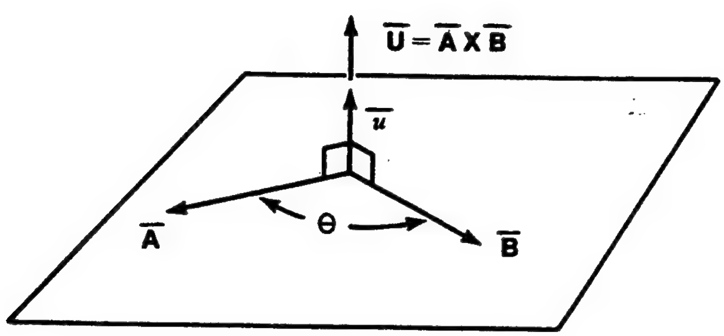



FIGURE 2.9. GEOMETRIC DEFINITION OF THE CROSS PRODUCT

The cross product vector $\bar{U}$ can be represented as

$$\bar{U} = \bar{A} \times \bar{B} = \left|\bar{A}\right| \left|\bar{B}\right| \sin \Theta \, \bar{u}$$

where $\bar{u}$ is a unit vector in the direction of $\bar{U}$, which is perpendicular to the plane containing $\bar{A}$ and $\bar{B}$.

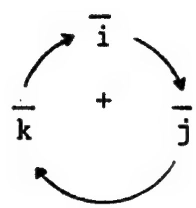
Some practical applications of the above definitions using the sine of zero and 90° are shown for unit vectors of a rectangular coordinate system.

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = \bar{0} \text{ (note the zero vector has any direction)}$$

$$\bar{i} \times \bar{j} = \bar{k} \text{ and } \bar{j} \times \bar{k} = \bar{i} \text{ and } \bar{k} \times \bar{i} = \bar{j} \text{ and}$$

$$\bar{j} \times \bar{i} = -\bar{k} \text{ and } \bar{k} \times \bar{j} = -\bar{i} \text{ and } \bar{i} \times \bar{k} = -\bar{j}$$

These cross products are used often, and an easy way to remember them is to use the aid




where the cross product in the positive direction from $\bar{i}$ to $\bar{j}$ gives a positive $\bar{k}$, and to reverse the direction gives a negative answer.

### 2.4.5 Cross Product Laws

If $\bar{A}$, $\bar{B}$, and $\bar{C}$ are vectors and m is a scalar, then

| | | |
|---|---|---|
| 1. | $\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$ | Anti-Commutative Product |
| 2. | $\bar{A} \times (\bar{B} + \bar{C}) = \bar{A} \times \bar{B} + \bar{A} \times \bar{C}$ | Distributive Product |
| 3. | $m(\bar{A} \times \bar{B}) = (m\bar{A}) \times \bar{B} = \bar{A} \times (m\bar{B})$ | Associative Product |

### 2.4.6 Vector Differentiation

The following treatment of vector differentiation has notation consistent with later courses and has been highly specialized for the USAF Test Pilot School curriculum. The scalar definition of the time derivative of a scalar function of the variable t is defined as,

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \left[ \frac{f(t + \Delta t) - f(t)}{\Delta t} \right]$$

Before proceeding, a vector function is defined as

$$F(t) = f_x(t)\bar{i} + f_y(t)\bar{j} + f_z(t)\bar{k},$$

where $f_x$, $f_y$, and $f_z$ are scalar functions of time and $\bar{i}$, $\bar{j}$, and $\bar{k}$ are unit vectors parallel to the x, y, and z axes, respectively. A vector function is a vector that changes magnitude and direction as a function of time and is referred to as a position vector. It gives the position of a particle in space at time t. The trace of the end points of the position vector gives the trajectory of the particle. The time derivative of a vector function with respect to some reference frame is defined as,

$$\frac{d\bar{F}(t)}{dt} = \lim_{\Delta t \to 0} \left[ \frac{\bar{F}(t + \Delta t) - \bar{F}(t)}{\Delta t} \right] =$$

$$\frac{df_x(t)}{dt}\bar{i} + \frac{df_y(t)}{dt}\bar{j} + \frac{df_z(t)}{dt}\bar{k} = \frac{df_x}{dt}\bar{i} + \frac{df_y}{dt}\bar{j} + \frac{df_z}{dt}\bar{k} =$$

$$\dot{f}_x\bar{i} + \dot{f}_y\bar{j} + \dot{f}_z\bar{k}$$

where the lack of a function variable indicates the function has the same variable as the differentiation variable, and the dot denotes time differentiation.

### 2.4.7 Vector Differentiation Laws

For vector functions $\bar{A}(t)$ and $\bar{B}(t)$, and scalar function $f(t)$

| | | |
|---|---|---|
| 1. | $\frac{d(\bar{A} + \bar{B})}{dt} = \frac{d\bar{A}}{dt} + \frac{d\bar{B}}{dt}$ | Distributive Derivative |
| 2. | $\frac{d(\bar{A} \bullet \bar{B})}{dt} = \bar{A} \bullet \frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \bullet \bar{B}$ | Dot Product Derivative |
| 3. | $\frac{d(\bar{A} \times \bar{B})}{dt} = \bar{A} \times \frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \times \bar{B}$ | Cross Product Derivative |
| 4. | $\frac{d}{dt}[f(t)\bar{B}] = f(t)\frac{d\bar{B}}{dt} + \frac{df(t)}{dt}\bar{B}$ | Scalar, Vector Product Derivative |

## 2.5 LINEAR VELOCITY AND ACCELERATION

The time derivative of a position vector relative to some reference system is the linear velocity. Note in particular that the velocity of a particle is a vector that has a direction and a magnitude. The magnitude of the velocity is referred to as speed. The second derivative is the linear acceleration.

Graphically, the derivative of a vector is illustrated as shown in Figure 2.10.

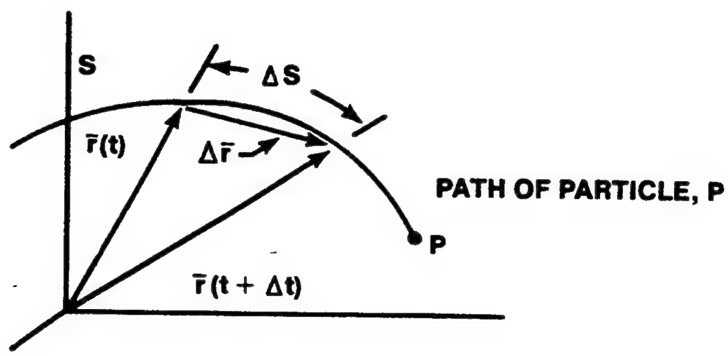



FIGURE 2.10. ILLUSTRATION OF THE DERIVATIVE OF A POSITION VECTOR

The difference between position vectors $\bar{r}(t + \Delta t)$ and $\bar{r}(t)$ is the numerator of the definition of the derivative. The arc length of the trajectory for some $\Delta t$ is $\Delta s$. If we neglect the division by $\Delta t$ and are concerned only with direction of the derivative, the difference of the two vectors is just $\Delta r$ which would have the direction as shown in Figure 2.10. The derivative for a vector $\bar{r}(t)$ can be expanded by multiplying by the quantity $\Delta s/\Delta s = 1$, as follows,

$$\frac{d\bar{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} \frac{\Delta s}{\Delta s} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta s} \frac{\Delta s}{\Delta t}$$

but as $\Delta t \to 0$, $|\Delta \bar{r}| = \Delta s$, therefore $\lim_{\Delta t \to 0} \Delta \bar{r}/\Delta s = \hat{e}_t$, since its magnitude is one

and its direction becomes tangential to the trajectory. The $\Delta s/\Delta t$ portion gives the magnitude of the derivative which should be noted as the speed of the particle or a change of distance per time. In summary, the first derivative of a vector function is tangential to the trajectory and has a magnitude that is the speed of the particle.

Using differentiation law four to take the derivative of the vector written in the form of magnitude times a unit vector,

$\bar{r}(t) = r(t)\hat{r}$, as follows,

$$\frac{d\bar{r}(t)}{dt} = \frac{d[r(t)\hat{r}]}{dt} = \frac{dr(t)}{dt}\hat{r} + r(t)\frac{d\hat{r}}{dt}$$

note that the linear velocity using this form of a vector has two components, the first is the rate of change of the scalar function with direction the same as the original vector itself. The second component is the scalar function itself with the rate of change of the unit vector as its direction. We know that the unit vector doesn't change magnitude, but it may change direction giving a non-zero derivative. In the development of the derivative earlier, this was overlooked since the rate of change of the $\bar{i}$, $\bar{j}$, and $\bar{k}$ vectors that are fixed in a coordinate system do not change direction or magnitude.

## 2.6 REFERENCE SYSTEMS

Linear velocity and acceleration have meaning only if expressed (or implied) in reference to another point and only if relative to a particular frame of reference. In this text for discussions of single reference systems, the linear velocity and acceleration will always be relative to the origin of the reference frame in which the problem is given and will be denoted by single letters, $\bar{V}$ and $\bar{a}$. If there are two reference systems in the problem, the notation will be changed to read

$$\bar{V}_{A/B}$$

which means the velocity of point or reference A relative to reference B. To take a time derivative of a vector relative to reference system "A," the notation will be

$$\frac{{}^{A}d\overline{F}}{dt}$$

There should be less confusion in multiple reference system problems concerning which reference frame the derivative is taken by using this notation. By introducing the concept of multiple reference systems, it is appropriate to discuss the chain rule. For two reference systems, the chain rule is simply stated. For point A in reference system B, which in turn is in another reference system C, the velocity of A relative to C is equal to

$$\overline{V}_{A/C} = \overline{V}_{A/B} + \overline{V}_{B/C} \tag{2.1}$$

While calculating derivatives when given the time function of the trajectory is seemingly simple, at times the derivatives may be difficult. Also, if the function is not known, the measurements available to determine the trajectory may be in terms of translational or rotational parameters which don't always lend themselves directly to a time function. Another method of determining velocities and accelerations will be determined using pure translation and rotation. Simplification will consist of very specific problems with convenient alignment of reference systems at specific instances in time. So it will appear that the time element has disappeared in the following analysis since the vectors will be constants at the instant we observe them.

## 2.7 DIFFERENTIATION OF A VECTOR IN A RIGID BODY

The two basic motions, translation and rotation, will be applied to a rigid body which is assumed not to bend or twist (every point in the body remains an equidistance from all others). It will become important to determine not only the velocity and acceleration of a point in a rigid body, but also that of a vector which lies in a rigid body.

### 2.7.1 Translation

If a body moves so that all the particles have the same velocity relative to some reference at any instant of time, the body is said to be

in pure translation. A vector in pure translation changes neither its magnitude nor direction while translating, so its first derivative would be zero. An example would be a vector from the center of gravity to the wingtip of an airplane in straight and level, unaccelerated flight with respect to a reference system attached to the earth's surface. From the ground it changes neither magnitude nor direction, although every point on the aircraft is traveling at the same velocity. See Figure 2.11.

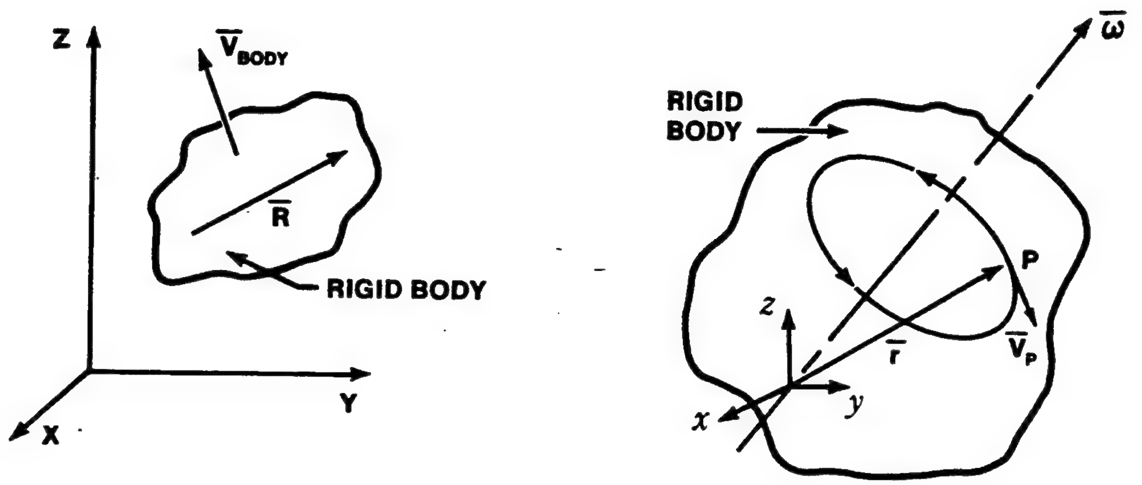



FIGURE 2.11. TRANSLATION AND ROTATION OF VECTORS IN RIGID BODIES

2.7.2 Rotation

If a body moves so that the particles along some line in the body have a zero velocity relative to some reference, the body is said to be in pure rotation relative to this reference. The line of stationary particles shown in Figure 2.11 is called the axis of rotation. A free vector that describes the rotation is called the angular velocity, $\overline{\omega}$, and has direction determined by the axis of rotation, using the right-hand rule to determine the sense. The chain rule as described for linear velocity applies to the angular velocity, as does a definition of its magnitude being angular speed. The first derivative of the angular velocity is the angular acceleration.

It can be proven that the linear velocity $\overline{V}$ of any point in a rigid body described by position vector $\overline{r}$ whose origin is along the axis of rotation can be written

$$\dot{\bar{r}} = \bar{V} = \bar{\omega} \times \bar{r} \tag{2.2}$$

Note the conventions using the right-hand rule apply, and $\bar{V}$ is perpendicular to the plane of $\bar{r}$ and $\bar{\omega}$.

The pure rotation of one reference system with respect to another would require a transformation of unit vectors from one system to another, unless the reference systems were conveniently aligned at the instant in question. Such transformations are considered beyond the scope of this course.

Equation 2.2 can be generalized to include any vector in a rigid body with pure rotation. Refer to Figure 2.12.

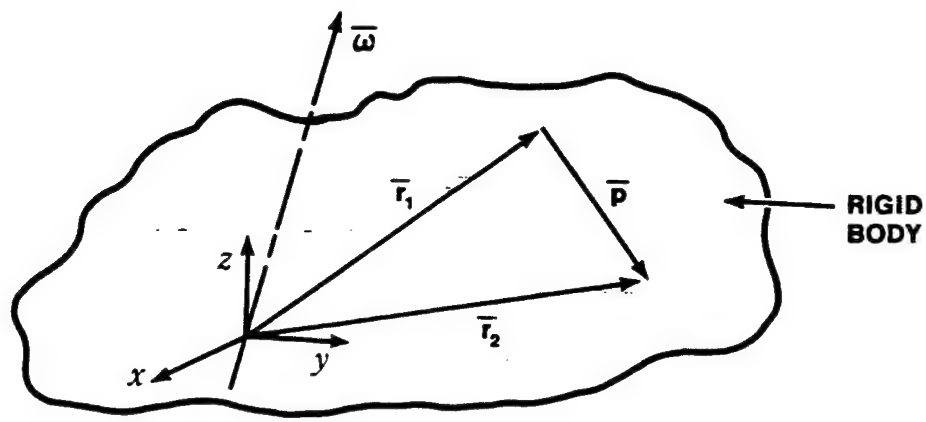



FIGURE 2.12. DIFFERENTIATION OF A FIXED VECTOR

Let $\bar{p}$ be a vector fixed anywhere in the rotating rigid body shown in Figure 2.12. The problem is to find the time rate of change of the vector. Two position vectors, $\bar{r}_1$ and $\bar{r}_2$, from the origin to the end points of the vector $\bar{p}$ are drawn. From vector addition

$$\bar{r}_1 + \bar{p} = \bar{r}_2$$

or solving

$$\bar{p} = \bar{r}_2 - \bar{r}_1$$

Differentiating

$$\dot{\bar{p}} = \dot{\bar{r}}_2 - \dot{\bar{r}}_1$$

From Equation 2.2,

$$\dot{\bar{r}}_2 = \bar{\omega} \times \bar{r}_2$$

and

$$\dot{\bar{r}}_1 = \bar{\omega} \times \bar{r}_1$$

so

$$\dot{\bar{p}} = \bar{\omega} \times \bar{r}_2 - \bar{\omega} \times \bar{r}_1$$

Since the cross product is distributive, this equation becomes

$$\dot{\bar{p}} = \bar{\omega} \times (\bar{r}_2 - \bar{r}_1) = \bar{\omega} \times \bar{p} \tag{2.3}$$

Therefore, the derivative of any fixed vector in a purely rotating rigid body is represented by the cross product of the angular velocity of the rotating body and the fixed vector.

### 2.7.3 Combination of Translation and Rotation in One Reference System

It is possible to combine the two types of velocity. An important point to notice here is that the velocities and accelerations are arrived at directly without the use of position vectors.

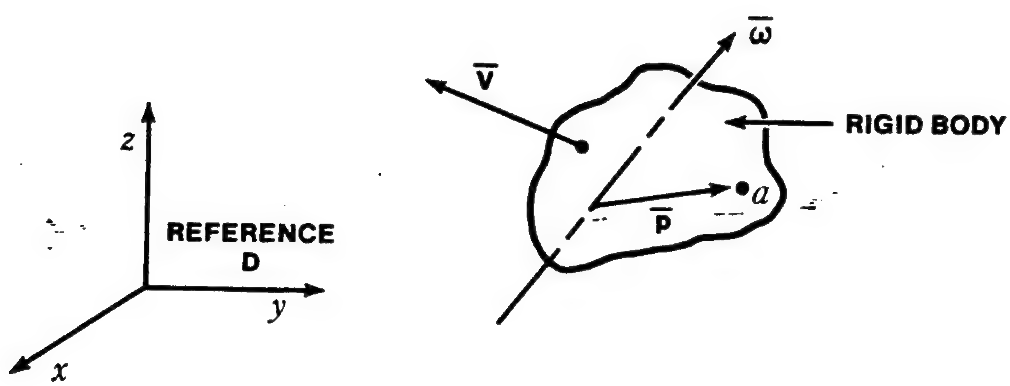



FIGURE 2.13. RIGID BODY IN TRANSLATION AND ROTATION

The velocity of point a in reference system D, Figure 2.13, will be calculated. The rigid body has a pure angular velocity, $\bar{\omega}$, and a pure translation, $\bar{v}$, in reference frame D. The requested velocity is just the sum,

$$\bar{V} = \bar{V}_{rotation} + \bar{V}_{translation}$$

$\bar{V}_{rotation}$ is equal to $\bar{\omega} \times \bar{p}$. $\bar{V}_{translation}$ is given as $\bar{v}$, so

$$\bar{V} = \bar{\omega} \times \bar{p} + \bar{v}$$

When working in one reference system, the acceleration may be calculated by taking the derivative of the velocity.

$$\bar{A} = \frac{d\bar{V}}{dt} = \frac{d(\bar{\omega} \times \bar{p})}{dt} + \frac{d\bar{v}}{dt} = \bar{\omega} \times \dot{\bar{p}} + \dot{\bar{\omega}} \times \bar{p} + \dot{\bar{v}}$$

Here, the $\dot{\bar{p}}$ is equal to $\bar{\omega} \times \bar{p}$ as was shown in Equation 2.3 and $\dot{\bar{v}}$ is the translational acceleration $\bar{a}$. The angular acceleration $\dot{\bar{\omega}}$ will not receive any special notation in this text.

So, the acceleration in a single reference system can be written

$$\bar{A} = \bar{\omega} \times (\bar{\omega} \times \bar{p}) + \dot{\bar{\omega}} \times \bar{p} + \bar{a}$$

### 2.7.4 Vector Derivatives in Different Reference Systems

The more general problem of relative motion between a point and a reference system that is itself moving relative to another reference system will be approached. More than one reference system is often used in order to simplify the analysis of general problems. As a first step, it is necessary to examine the procedure of differentiation with respect to time in the presence of two references moving relative to each other.

A reference system is a non-deformable system and may be considered a rigid body. So, the work done so far applies here. Figure 2.14 gives the vectors used in the following analysis.

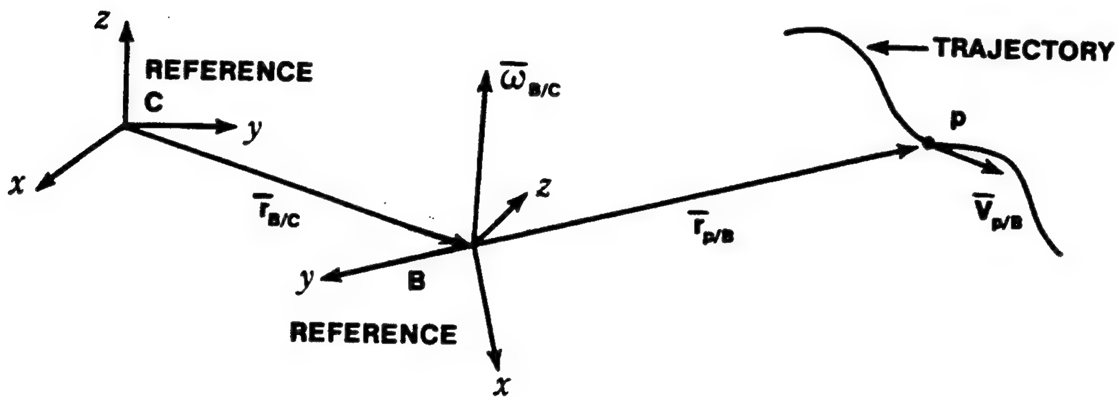



FIGURE 2.14. MOTION WITH TWO REFERENCE SYSTEMS

The problem above shows point p with position vector $\overline{r}_{p/B}$, moving with respect to the reference B, and the origin of B with position vector $\overline{r}_{B/C}$, moving with respect to reference system C. The reference system B also has an angular velocity with respect to C of $\overline{\omega}_{B/C}$. The goal of the following development will be to find the time rate of change of the position vector in the B frame as seen from the C frame or notationally

$$\frac{{}^{C}d}{dt}\,\overline{r}_{p/B}$$

It is very important to realize that this is not the same as the velocity of the point as seen from the C frame. Rather it is the rate of change of a position vector in one frame as seen from another frame. So the derivative sought is not $\overline{V}_{p/C}$. This velocity would be obtained by using the chain rule as given in Equation 2.1.

A representative example is the motion of a point on an aircraft with a body axis system at the center of gravity and the aircraft moving along some path relative to the ground. The second reference system is attached to the ground. It will be assumed in this analysis that the two reference systems have the same unit vectors. Careful attention will be given to circumstances resulting from axes that may not be conveniently aligned during the analysis.

Beginning with the position vector in the frame B,

$$\bar{r}_{p/B} = x\bar{i} + y\bar{j} + z\bar{k}$$

Differentiating this vector with respect to time relative to the C reference frame presents a problem since the unit vectors of the B system are rotating as seen in the C system.

So, the derivative must be done in two parts using the fourth law given earlier.

$$\frac{^C d}{dt}\bar{r}_{p/B} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x\dot{\bar{i}} + y\dot{\bar{j}} + z\dot{\bar{k}}$$

But the unit vector's derivatives can be written as vector in a single reference system with derivatives as seen in Equation 2.3. Thus,

$$\dot{\bar{i}} = \bar{\omega}_{B/C} \times \bar{i}, \text{ etc.}$$

So

$$\frac{^C d}{dt}\bar{r}_{p/B} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x\dot{\bar{i}} + y\dot{\bar{j}} + z\dot{\bar{k}}$$

$$= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + x(\bar{\omega}_{B/C} \times \bar{i}) + y(\bar{\omega}_{B/C} \times \bar{j}) + z(\bar{\omega}_{B/C} \times \bar{k})$$

$$= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega}_{B/C} \times (x\bar{i}) + \bar{\omega}_{B/C} \times (y\bar{j}) + \bar{\omega}_{B/C} \times (z\bar{k})$$

$$= \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k} + \bar{\omega}_{B/C} \times (x\bar{i} + y\bar{j} + z\bar{k})$$

The first three terms are recognized as the velocity of p in the B system and the next term is the cross product of the angular velocity of the B system with respect to the C system and the position vector in the B system.

$$\frac{^C d}{dt}\bar{r}_{p/B} = \dot{\bar{r}}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B} = \bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B} \qquad (2.4)$$

This equation may be generalized to any vector in one reference system relative to another. This is a very important relationship and will be used in Chapter 4, in the derivation of the aircraft equations of motion.

The acceleration of a particle at point p would be handled using the definition of acceleration.

$$\bar{A}_{p/C} = \frac{{}^{C}d}{dt} \bar{V}_{p/C} \qquad (2.5)$$

Note Equation 2.5 does not address $\frac{{}^{C}d}{dt} \bar{V}_{p/B}$

Hopefully, the velocity would be written in a simple form allowing simple differentiation to obtain the acceleration. If not, a simple exchange of notation with Equation 2.4 would be necessary.

COMMENT The material presented thus far is sufficient to enable solution of any linear or angular velocity or acceleration in a kinematics problem. However, another analysis follows which may clarify multi-reference problems and will provide definition of some terms that will be of value in later courses.

2.7.4.1 Transport Velocity. In this analysis of motion relative to two reference systems, a different approach is taken to the problem. Figure 2.14 is expanded as shown in Figure 2.15 to include the position vector directly from reference C to the point p.

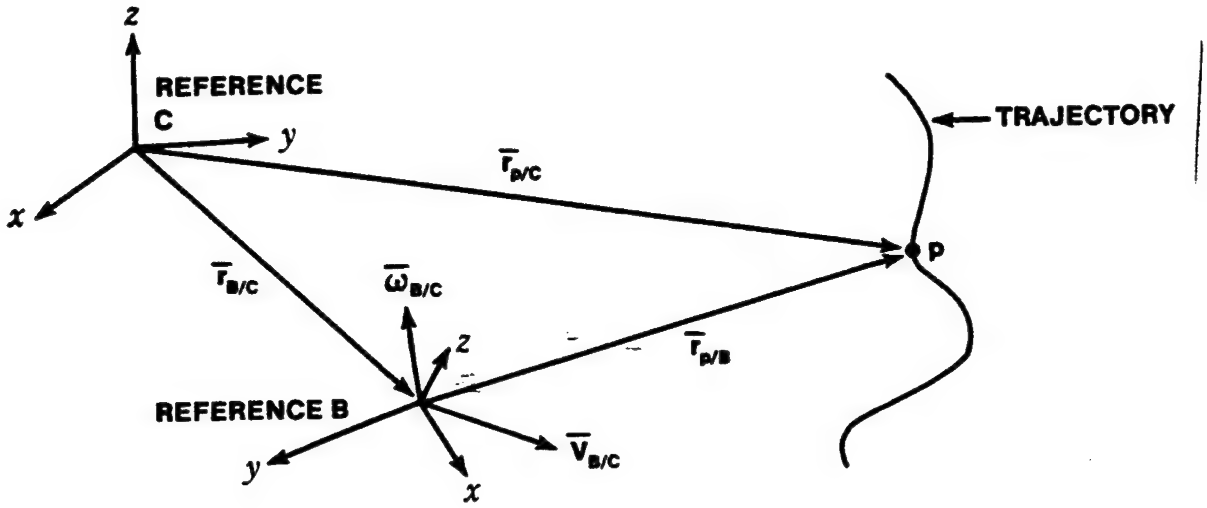



FIGURE 2.15. TWO REFERENCE SYSTEM VECTORS

Thus

$$\overline{r}_{p/C} = \overline{r}_{p/B} + \overline{r}_{B/C}$$

and

$$\overline{V}_{p/C} = \dot{\overline{r}}_{p/C} = \frac{{}^{C}d}{dt}\,\overline{r}_{p/C} = \frac{{}^{C}d}{dt}\,\overline{r}_{p/B} + \frac{{}^{C}d}{dt}\,\overline{r}_{B/C}$$

where the first term is Equation 2.4 and the second is $\overline{V}_{B/C}$. Substituting these terms,

$$\overline{V}_{p/C} = \dot{\overline{r}}_{p/C} = \overline{V}_{p/B} + \overline{\omega}_{B/C} \times \overline{r}_{p/B} + \overline{V}_{B/C} \qquad (2.6)$$

or

$$\overline{V}_{p/C} = \overline{V}_{p/B} + \overline{V}_{pTC}$$

where

$$\overline{V}_{pTC} = \overline{\omega}_{B/C} \times \overline{r}_{p/B} + \overline{V}_{B/C}$$

This term is called the transport velocity. The interpretation of transport velocity defined in this equation is such that $\overline{V}_{p/B}$ is still the velocity of p relative to B and $\overline{V}_{pTC}$ is the velocity in C that p would have,

if p were fixed in B. Note this is just the sum of the translation and rotation of frame B relative to frame C if the point p is considered fixed.

2.7.4.2 Special Acceleration. By taking the derivative of the velocity, as in Equation 2.5, and applying the distributive law to the cross product,

$$\bar{A}_{p/C} = \frac{{}^{C}d}{dt}\bar{V}_{p/C} = \frac{{}^{C}d}{dt}\bar{V}_{p/B} + \frac{{}^{C}d}{dt}\bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \frac{{}^{C}d}{dt}\bar{r}_{p/B} + \frac{{}^{C}d}{dt}\bar{V}_{B/C}$$

Now, substituting with the notation for acceleration where possible,

$$\bar{A}_{p/C} = \frac{{}^{C}d}{dt}\bar{V}_{p/B} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \frac{{}^{C}d}{dt}\bar{r}_{p/B} + \bar{A}_{B/C}$$

The two remaining terms with derivative notation should be recognized as applications of Equation 2.4. So, substituting

$$\bar{A}_{p/C} = (\dot{\bar{V}}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B}) + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times (\bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B}) + \bar{A}_{B/C}$$

Expanding and noting

$$\dot{\bar{V}}_{p/B} = \bar{A}_{p/B}$$

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B}) + \bar{A}_{B/C}$$

Rearranging and combining the two like terms

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + 2\bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B}) \qquad (2.7)$$

Of the five terms remaining in the acceleration equation, the last two have descriptive names.

$2\bar{\omega}_{B/C} \times \bar{V}_{p/B}$ is called the Coriolis acceleration, and

$\bar{\omega}_{B/C} \times \bar{\omega}_{B/C} \times \bar{R}_{p/B}$ is called the Centripetal acceleration.

The terms in Equation 2.7 that are independent of the motion of p relative to frame B are called the transport acceleration. These terms

provide the acceleration in frame C of a point that is fixed at p at the instant in question. Notationally, the transport acceleration is

$$\bar{A}_{pTC} = \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B})$$

These concepts are difficult to realize until a few problems are attempted.

2.7.4.3 <u>Example Two Reference System Problem</u>. The angular velocity of the

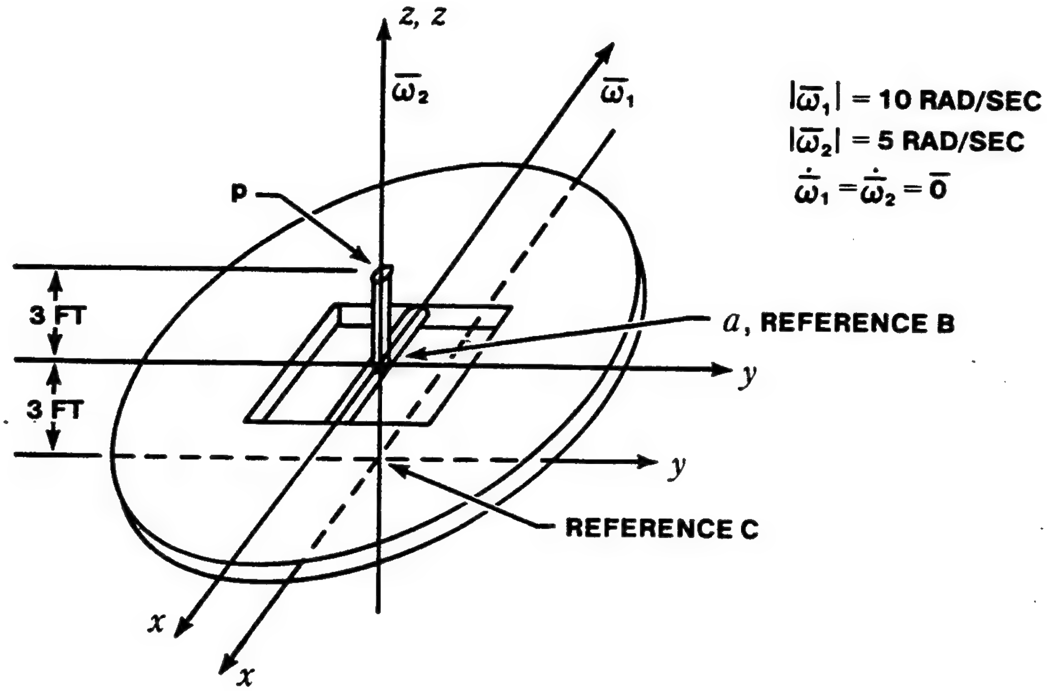



FIGURE 2.16. TWO REFERENCE SYSTEM PROBLEM

arm ap relative to the disk in Figure 2.16 is 10 rad/sec, shown vectorally in the diagram as $\bar{\omega}_1$, while the angular velocity of the disk relative to the ground is 5 rad/sec, shown vectorally as $\bar{\omega}_2$. The angular accelerations are zero. Reference B is attached to the platform, while frame C is fixed to the ground, three feet below the disk. At the instant in question, the arm ap is in the vertical position, and the reference axes directions coincide, although displaced.

Find the velocity and acceleration of point p relative to the fixed reference frame C.

Using Equation 2.6

$$\bar{V}_{p/C} = \dot{\bar{r}}_{p/C} = \bar{V}_{p/B} + \bar{\omega}_{B/C} \times \bar{r}_{p/B} + \bar{V}_{B/C} \qquad (2.6)$$

we know the last term, $\bar{V}_{B/C} = \bar{0}$, since the B frame is only rotating relative to C.

$$\bar{\omega}_{B/C} = \bar{\omega}_2 = 5\bar{k} \text{ rad/sec and } \bar{r}_{p/B} = 3\bar{k} \text{ feet, by observation}$$

This leaves $\bar{V}_{p/B}$ which involves angular velocity $\bar{\omega}_1 = -10\bar{i}$, relative to B.

$$\bar{V}_{p/B} = \bar{\omega}_1 \times \bar{r}_{p/B} = (-10\bar{i} \times 3\bar{k}) = (-30)(-\bar{j}) = 30\bar{j} \text{ ft/sec}$$

Substituting all the parts into Equation 2.6

$$\bar{V}_{p/C} = 30\bar{j} + 5\bar{k} \times 3\bar{k} + 0 = 30\bar{j} + 15(\bar{k} \times \bar{k}) = \underline{30\bar{j} \text{ ft/sec}}$$

For the acceleration, the general expression is Equation 2.7

$$\bar{A}_{p/C} = \bar{A}_{p/B} + \bar{A}_{B/C} + \dot{\bar{\omega}}_{B/C} \times \bar{r}_{p/B} + 2\bar{\omega}_{B/C} \times \bar{V}_{p/B} + \bar{\omega}_{B/C} \times (\bar{\omega}_{B/C} \times \bar{r}_{p/B})$$

The only unknown terms are $\bar{A}_{B/C} = \bar{0}$ and $\bar{A}_{p/B}$. The latter is a centripetal acceleration due to the rotation of the arm. The centripetal acceleration may be arrived at in several different ways,

$$\bar{A}_{p/B} = \frac{{}^{B}d}{dt}\bar{V}_{p/B} = \frac{d}{dt}(\bar{\omega}_1 \times \bar{r}_{p/B}) = \dot{\bar{\omega}}_1 \times \bar{r}_{p/B} + \bar{\omega}_1 \times \dot{\bar{r}}_{p/B}$$

$$= \bar{0} + \bar{\omega}_1 \times (\bar{\omega}_1 \times \bar{r}_{p/B}) = (-10\bar{i}) \times (30\bar{j})$$

$$= \underline{-300\bar{k} \text{ ft/sec}^2}$$

Substituting this value and the others already calculated

$$\bar{A}_{p/c} = -300\bar{k} + \bar{0} + \bar{0} \times 3\bar{k} + 2(5\bar{k} \times 30\bar{j}) + 5\bar{k} \times (5\bar{k} \times 3\bar{k}) = \underline{-300\bar{i} - 300\bar{k}}$$

While working problems where there is a choice of axes, be careful to choose so that as many parameters as possible are equal to zero, and most importantly so that the axes are aligned at the instant in question. Also, whether a reference system is fixed in a body or not will have profound effects on the velocities as seen from that origin. Try to place yourself at the origin of a system and visualize the velocity and acceleration seen to help avoid confusion. Also check your answers to see if they are logical, both in magnitude and direction. The right-hand rule is essential.

When working with large systems, with many variables it becomes necessary to develop a shorthand method of writing systems of equations. The development of matrix algebra is the solution.

## 2.8 MATRICES

An m x n matrix is a rectangular array of quantities arranged in m rows and n columns. When there is no possibility of confusion, matrices are often represented by single capital letters. More commonly, however, they are represented by displaying the quantities between brackets; thus,

$$A = [A]_{mxn} = \|a_{ij}\|_{mxn} = [a_{ij}]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Note that $a_{ij}$ refers to the element in the ith row and jth column of [A]. Thus, $a_{23}$ is the element in the second row and third column. Matrices having only one column (or one row) are called column (or row) vectors. The matrix [X] below is a column vector, and the matrix [Y] is a row vector.

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad [Y] = [y_1, y_2 \ \cdots \ y_n]$$

A matrix, unlike the determinant, is not assigned any "value"; it is simply an array of quantities. Matrices may be considered as single algebraic entities and combined (added, subtracted, multiplied) in a manner similar to the combination of ordinary numbers. It is necessary, however, to observe specialized algebraic rules for combining matrices. These rules are somewhat more complicated than for "ordinary" algebra. The effort required to learn the rules of matrix algebra is well justified, however, by the simplification and organization which matrices bring to problems in linear algebra.

### 2.8.1 Matrix Equality

Two matrices $[A] = [a_{ij}]$ and $[B] = [b_{ij}]$ are equal if and only if they are identical; i.e., if and only if they contain the same number of rows and the same number of columns, and $a_{ij} = b_{ij}$ for all values of i and j. Thus, the statement

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 19 \end{bmatrix}$$

is equivalent to the statements

$$\begin{aligned} a_{11} &= 2 \\ a_{12} &= 4 \\ &\vdots \\ &\text{etc.} \end{aligned}$$

### 2.8.2 Matrix Addition

Two matrices having the same number of rows and the same number of columns are defined as being conformable for addition and may be added by adding corresponding elements; i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots \\ a_{21} + b_{21} & & \text{etc.} \\ \vdots & & \end{bmatrix}$$

Thus

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 6 \\ -3 & 0 \end{bmatrix}$$

### 2.8.3 Matrix Multiplication by a Scalar

A scalar is a single number. A matrix of any shape may be multiplied by a scalar by multiplying each element of the matrix by the scalar. That is:

$$k[A] = k \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots \\ ka_{21} & & \\ \vdots & & \end{bmatrix}$$

For example,

$$3 \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 9 & 0 \end{bmatrix}$$

### 2.8.4 Matrix Multiplication

Matrix multiplication can be defined for any two matrices when the number of columns of the first is equal to the number of rows of the second matrix. This can be stated mathematically as:

$$[A]_{i \times m}\;[B]_{m \times j} = [C]_{i \times j}$$

where

$$c_{ij} = \sum_{k=1}^{m} a_{ik}\;b_{kj}$$

Multiplication is not defined for other matrices.

Equation 2.8 demonstrates the product of two, 2 x 2 matrices.

$$[A]_{2\times2}\;[B]_{2\times2} = [C]_{2\times2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

or using the definition of multiplication,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\;b_{11} + a_{12}\;b_{21} & a_{11}\;b_{12} + a_{12}\;b_{22} \\ a_{21}\;b_{11} + a_{22}\;b_{21} & a_{21}\;b_{12} + a_{22}\;b_{22} \end{bmatrix} \quad (2.8)$$

This situation is sufficiently general to point the way to an orderly multiplication process for matrices of any order.

In the indicated product,

$$[A]\;[B] = [C]$$

the left-hand factor may be treated as a bundle of row-vectors,

$$[A] = \begin{bmatrix} [a_{11} & a_{12}] \\ [a_{21} & a_{22}] \end{bmatrix}$$

and the right-hand factor as a bundle of column vectors,

$$[B] = \begin{bmatrix} \begin{bmatrix} b_{11} \\ \\ b_{21} \end{bmatrix} & \begin{bmatrix} b_{12} \\ \\ b_{22} \end{bmatrix} \end{bmatrix}$$

Then,

$$\begin{bmatrix} [a_{11} & a_{12}] \\ \\ [a_{21} & a_{22}] \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} b_{11} \\ \\ b_{21} \end{bmatrix} & \begin{bmatrix} b_{12} \\ \\ b_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ \\ c_{21} & c_{22} \end{bmatrix} \qquad (2.9)$$

and by comparison with Equation 2.8

$$\begin{bmatrix} [a_{11} & a_{12}] \\ [a_{21} & a_{22}] \end{bmatrix} \begin{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [a_{11} \quad a_{12}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{11} \quad a_{12}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \\ [a_{21} \quad a_{22}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{21} \quad a_{22}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \end{bmatrix} \qquad (2.10)$$

where, by definition,

$$[a_{11} \quad a_{12}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = [a_{11} \; b_{11} + a_{12} \; b_{21}]$$

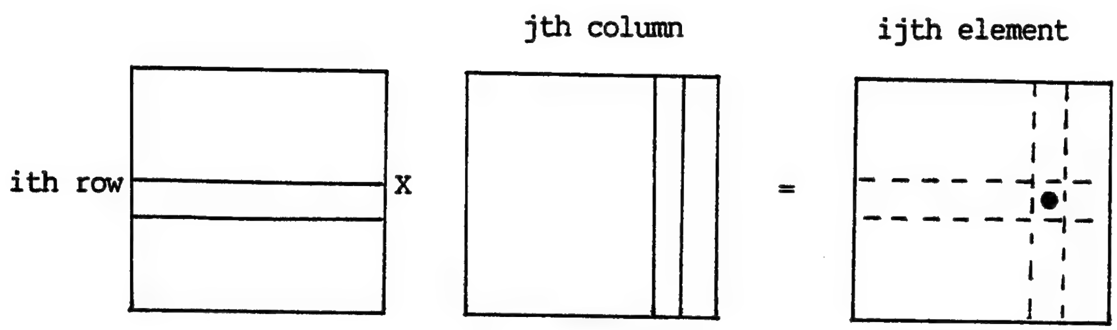
$$[a_{11} \quad a_{12}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = [a_{11} \; b_{12} + a_{12} \; b_{22}]$$

etc.

A comparison of Equations 2.9 and 2.10 shows that if the rows of [A] and the columns of [B] are treated as vectors, then $c_{ij}$ in the product [C] = [A] [B] is the dot product of the ith row of [A] and the jth column of [B]. This rule holds for matrices of any size, i.e.,

$$c_{ij} = [a_{i1}\ a_{i2}\ \dots\ a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \left[a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \dots + a_{in}\, b_{nj}\right]$$

Matrix multiplication is therefore a "row-on-column" process:




$$\begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} [3 \quad 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [3 \quad 2] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ [-1 \quad 4] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [-1 \quad 4] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ [0 \quad 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} & [0 \quad 2] \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 \\ -5 & -2 \\ -2 & 0 \end{bmatrix}$$

The indicated product [A] [B] can be carried out only if [A] and [B] are conformable; that is, for conformability in multiplication, the number of columns in [A] must equal the number of rows in [B]. For example, the expression

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

is meaningless (as an attempt to carry out the multiplication will show) because the number of columns in [A] is two and the number of rows in [B] is three. A convenient rule is this: if [A] is an m x n matrix (m rows, n columns) and [B] is an n x p matrix, then [C] = [A] [B] is an m x p matrix. That is,

$$[A]_{m \times n} \; [B]_{n \times p} = [C]_{m \times p}$$

Matrix algebra differs significantly from "ordinary" algebra in that multiplication is not commutative. In general, that is,

$$[A]\,[B] \neq [B]\,[A]$$

For example, if

$$[A] = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$$

then

$$[A]\,[B] = \begin{bmatrix} 4 & -6 \\ 4 & 0 \end{bmatrix}$$

$$[B]\,[A] = \begin{bmatrix} 2 & -5 \\ 4 & 2 \end{bmatrix}$$

Because multiplication is non-commutative, care must be taken in describing the product

$$[C] = [A]\,[B]$$

to say that [A] "premultiplies" [B], or, equivalently, that [B] "post-multiplies" [A].

### 2.8.5 The Identity Matrix

The identity (or unit) matrix [I] occupies the same position in matrix algebra that the number one does in ordinary algebra. That is, for any matrix [A],

$$[I]\,[A] = [A]\,[I] = [A]$$

The identity [I] is a square matrix consisting of ones on the principal ( $\longleftrightarrow$ ) diagonal and zeros everywhere else; i.e.,

$$[I] = \begin{bmatrix} 1 & 0 & 0 \ldots 0 \\ 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \\ 0 & 0 & 0 \ldots 1 \end{bmatrix}$$

The order (the number of rows and columns) of an identity matrix depends entirely on the requirement of conformability with adjacent matrices. For example, if

$$[A] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

then

$$[I]\,[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$[A]\,[I] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

Thus, the "left" identity for [A] is 2 x 2 and the "right" identity for [A] is 3 x 3; however, they both leave [A] unaltered.

### 2.8.6 The Transposed Matrix

The transpose of [A], labeled $[A]^T$, is formed by interchanging the rows and columns of [A]. That is,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

The jth row vector becomes the jth column vector, and vice versa. For example,

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5+j4 \\ -10 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -10 \\ 5+j4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

### 2.8.7 The Inverse Matrix

Matrix multiplication has been defined; it is natural to inquire next if there is some way to divide matrices. There is not, properly speaking, a division operation in matrix algebra; however, an equivalent result is obtained through the use of the inverse matrix.

In ordinary algebra, every number a (except zero) has a multiplicative inverse, $a^{-1}$ defined as follows: A quantity $a^{-1}$ is the inverse of a if

$$a \cdot a^{-1} = a^{-1} \quad a = 1$$

In the same way, the matrix $[A]^{-1}$ is called the inverse matrix of [A] if

$$[A]\ [A]^{-1} = [A]^{-1}\ [A] = [I]$$

The symbol 1/a is normally used to signify $a^{-1}$. Since ordinary multiplication is commutative,

$$(1/a) \cdot (b) = (b) \cdot (1/a) = b \div a$$

for any number b. The use of the division symbol (÷) in this instance is useful and unambiguous. In matrix algebra, however, multiplication is not commutative. Therefore,

$$[A]^{-1}\ [B] \neq [B]\ [A]^{-1}$$

and the expression

$$[B] \div [A]$$

cannot be used since it may have either of the (unequal) meanings in the previous equation. Instead of saying "divide [B] by [A]," one must say either "premultiply [B] by $[A]^{-1}$" or "postmultiply [B] by $[A]^{-1}$." The results, in general, are different.

2.8.8 Singular Matrices

Matrices which cannot be inverted are called singular. For inversion to be possible, a matrix must possess a determinant not equal to zero. For example, the matrix

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}$$

is singular because it is not square, and a determinant cannot be computed. The matrix

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

is singular because its determinant vanishes.

Matrices which do possess an inverse are called nonsingular.

## 2.9 SOLUTION OF LINEAR SYSTEMS

Consider the set of equations

$$\left.\begin{array}{ccccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & y_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}\,x_1 & + & a_{n2}\,x_2 & + & \cdots & + & a_{nn}\,x_n & = & y_n \end{array}\right\} \quad (2.11)$$

That is,

$$[A]\,[X] = [Y]$$

Assuming that the inverse of [A] has been computed, both sides of this equation may be premultiplied by $[A]^{-1}$, giving

$$[A]^{-1}\,[A]\,[X] = [A]^{-1}\,[Y]$$

From the definition of the inverse matrix,

$$[I]\,[X] = [A]^{-1}\,[Y]$$

from which, finally,

$$[X] = [A]^{-1}\,[Y]$$

Thus, the system of Equation 2.11 may be solved for $x_1, x_2, \ldots x_n$ by computing the inverse of [A].

### 2.9.1 Computing the Inverse

There is a straightforward four step method for computing the inverse of a given matrix [A]:

Step 1. Compute the determinant of [A]. This determinant is written as $|A|$. If the determinant is zero or does not exist, the matrix [A] is defined as singular and an inverse cannot be found.

Step 2. Transpose matrix [A]. The resulting matrix is written $[A]^T$.

Step 3. Replace each element $a_{ij}$ of the transposed matrix by its cofactor $A_{ij}$. This resulting matrix is defined as the adjoint of matrix [A] and is written: Adj [A].

Step 4. Divide the adjoint matrix by the scalar value of the determinant of [A] which was computed in Step 1. The resulting matrix is the inverse and is written: $[A]^{-1}$.

This procedure can be summarized as follows: To calculate the inverse of [A] calculate the adjoint of [A] and divide by the determinant of [A] or

$$[A]^{-1} = \frac{\text{Adj } [A]}{A}$$

Example: Find $[A]^{-1}$, if

$$[A] = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Step 1. Compute the determinant of [A]. Expanding about the first row

$$A = \begin{vmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

$$A = 3(-5 -2) -2(-1 + 0) + 0(2 - 0)$$

$$A = -21 + 2 + 0 = -19$$

The determinant has the value -19; therefore an inverse can be computed.

Step 2. Transpose [A]

$$[A]^T = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 3: Replace each element $a_{ij}$ of $[A]^T$ by its cofactor $A_{ij}$ to determine the adjoint matrix. Note that signs alternate from a positive $A_{11}$

$$\text{Adj }[A] = \begin{bmatrix} \begin{vmatrix} 5 & 2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}$$

Step 4: Divide by the scalar value of the determinant of [A] which was computed as -19 in Step 1.

$$[A]^{-1} = \frac{1}{-19} \begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}$$

### 2.9.2 Product Check

From the definition of the inverse matrix

$$[A]^{-1}\ [A] = [I]$$

This fact may be used to check a computed inverse. In the case just completed

$$[A]^{-1}\ [A] = \frac{1}{-19}\begin{bmatrix} -7 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & -6 & 13 \end{bmatrix}\begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$[A]^{-1}\ [A] = \frac{1}{-19}\begin{bmatrix} -19 & 0 & 0 \\ 0 & -19 & 0 \\ 0 & 0 & -19 \end{bmatrix}$$

$$[A]^{-1}\ [A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A]^{-1}\ [A] = [I]$$

Since the product does come out to be the identity matrix, the computation was correct.

2.9.3 Example Linear System Solution

Given the following set of simultaneous equations, solve for $x_1$, $x_2$, and $x_3$.

$$\left.\begin{aligned} 3x_1 + 2x_2 - 2x_3 &= y_1 \\ -x_1 + x_2 + 4x_3 &= y_2 \\ 2x_1 - 3x_2 + 4x_3 &= y_3 \end{aligned}\right\} \qquad (2.12)$$

This set of equations can be written as

$$[A]\ [X] = [Y]$$

or

$$[X] = [A]^{-1}\ [Y]$$

Thus, the system of Equations 2.12 can be solved for the values of $x_1$, $x_2$, and $x_3$ by computing the inverse of [A].

[A] [X] = [Y]

$$\begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Step 1: Compute the determinant of [A]. Expanding about the first row

$$A = 3\ (4 + 12)\ -2\ (-4\ -8)\ -2\ (3 - 2)$$

$$A = 48 + 24 - 2 = 70$$

Step 2: Transpose [A]

$$[A]^T = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -3 \\ -2 & 4 & 4 \end{bmatrix}$$

Step 3: Determine the adjoint matrix by replacing each element in $[A]^T$ by its cofactor

$$\text{Adj } [A] = \begin{bmatrix} \begin{vmatrix} 1 & -3 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$$

$$\text{Adj } [A] = \begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}$$

Step 4: Divide by the scalar value of the determinant of [A] which was computed as 70 in Step 1.

$$[A]^{-1} = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}$$

Product Check

$$[A]^{-1}\,[A] = [I]$$

$$[A]^{-1}\,[A] = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$[A]^{-1}\,[A] = \frac{1}{70}\begin{bmatrix} 70 & 0 & 0 \\ 0 & 70 & 0 \\ 0 & 0 & 70 \end{bmatrix} \tag{2.13}$$

$$[A]{-1}\,[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the product in Equation 2.13 is the identity matrix, the computation is correct. The values of $x_1$, $x_2$, and $x_3$ can now be found for any $y_1$, $y_2$, and $y_3$ by premultiplying [Y] by $[A]^{-1}$.

$$[X] = [A]^{-1}\,[Y]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{70}\begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For example, if $y_1 = 1$, $y_2 = 13$, and $y_3 = 8$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \\ 8 \end{bmatrix}$$

$$x_1 = \frac{1}{70}(16 - 26 + 80) = \frac{70}{70} = 1$$

$$x_2 = \frac{1}{70}(12 + 208 - 80) = \frac{140}{70} = 2$$

$$x_3 = \frac{1}{70}(1 + 169 + 40) = \frac{210}{70} = 3$$

Solution of sets of simultaneous equations using matrix algebra techniques has wide application in a variety of engineering problems.

## PROBLEMS

2.1 Is $\bar{V} = \frac{\sqrt{3}}{3}\bar{i} + \frac{\sqrt{3}}{3}\bar{j} + \frac{\sqrt{3}}{3}\bar{k}$ a unit vector?

2.2 Find a unit vector in the direction of

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$

2.3 Are the following two vectors equal?

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$

$$\bar{B} = 4\bar{i} + 6\bar{j} - 2\bar{k}$$

2.4 The following forces measured in pounds act on a body

$$\bar{F}_1 = 2\bar{i} + 3\bar{j} - 5\bar{k}$$

$$\bar{F}_2 = -5\bar{i} + \bar{j} + 3\bar{k}$$

$$\bar{F}_3 = \bar{i} - 2\bar{j} + 4\bar{k}$$

$$\bar{F}_4 = 4\bar{i} - 3\bar{j} - 2\bar{k}$$

Find the resultant force vector and the magnitude of the resultant force vector.

2.5 If

$$\bar{A} = 3\bar{i} - \bar{j} - 4\bar{k}$$

$$\bar{B} = -2\bar{i} + 4\bar{j} - 3\bar{k}$$

$$\bar{C} = \bar{i} + 2\bar{j} - \bar{k}$$

Find

$$2\bar{A} - \bar{B} + 3\bar{C} =$$

$$|\bar{A} + \bar{B} + \bar{C}| =$$

$$|3\bar{A} - 2\bar{B} + 4\bar{C}| =$$

2.6 The position vectors of points P and Q are given by

$$\bar{r}_1 = 2\bar{i} + 3\bar{j} - \bar{k}$$
$$\bar{r}_2 = 4\bar{i} - 3\bar{j} + 2\bar{k}$$

Determine the vector from P to Q ($\overline{PQ}$) and find its magnitude.

2.7 Find $\bar{A} \cdot \bar{B}$ using $\bar{A}$ and $\bar{B}$ from Problem 2.5.

2.8 Given

$$\bar{A} = 2\bar{i} + 3\bar{j} - \bar{k}$$
$$\bar{B} = 4\bar{i} + 6\bar{j} - 2\bar{k}$$

a. Find $\bar{A} \cdot \bar{B}$
b. Find the angle between $\bar{A}$ and $\bar{B}$.

2.9 Evaluate

$$\bar{j}\cdot(2\bar{i} - 3\bar{j} + \bar{k}) =$$
$$(2\bar{i}\cdot\bar{j})\cdot(3\bar{i} + \bar{k}) =$$

2.10 If

$$\bar{A} = 3\bar{i} - \bar{j} - 4\bar{k}$$
$$\bar{B} = -2\bar{i} + 4\bar{j} - 3\bar{k}$$

Find $\bar{A} \times \bar{B}$

2.11 Determine the value of "a" so that $\bar{A}$ and $\bar{B}$ below are perpendicular.

$$\bar{A} = 2\bar{i} + a\bar{j} + \bar{k}$$
$$\bar{B} = 4\bar{i} - 2\bar{j} - 2\bar{k}$$

2.12 Determine a unit vector perpendicular to the plane of A and B below.

$$\bar{A} = 2\bar{i} - 6\bar{j} - 3\bar{k}$$
$$\bar{B} = 4\bar{i} + 3\bar{j} - \bar{k}$$

2.13 If

$$\bar{A} = 2\bar{i} - 3\bar{j} - \bar{k}$$
$$\bar{B} = \bar{i} + 4\bar{j} - 2\bar{k}$$

Find

$$\bar{A} \times \bar{B}$$
$$\bar{B} \times \bar{A}$$

and

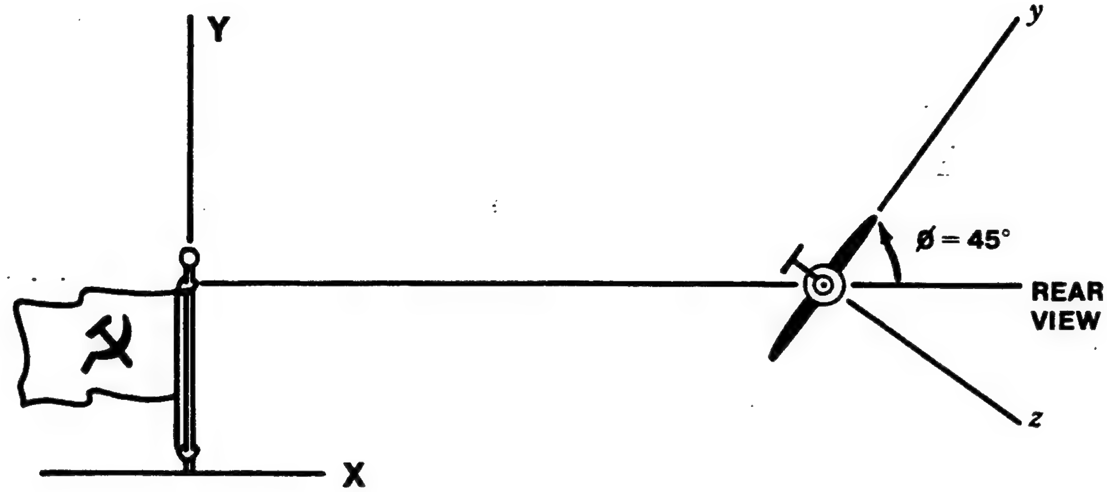
$(\bar{A} + \bar{B}) \times (\bar{A} - \bar{B})$ (the quick way using vector algebra).

2.14 Evaluate

a. $2\bar{j} \times (3\bar{i} - 4\bar{k})$

b. $(\bar{i} + 2\bar{j}) \times \bar{k}$

2.15 The aircraft shown below is flying around the flagpole in a steady state turn at a true velocity of 600 ft/sec. The turn radius is 6,000 ft. What is turn rate $\bar{\omega}$ expressed in unit vectors $(\bar{i}, \bar{j}, \bar{k})$ of the XYZ system shown?




2.16 For the same aircraft and conditions as Problem 2.15, what is turn rate expressed in unit vectors $(\bar{i}, \bar{j}, \bar{k})$ of the xyz system shown?

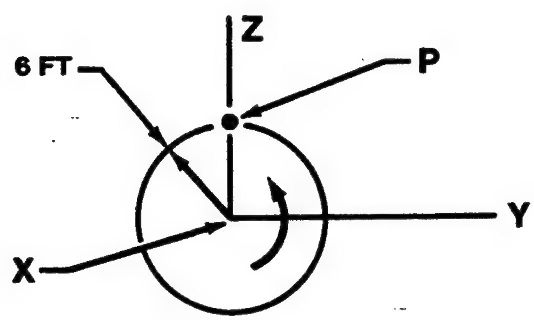
2.17 Given

$$\bar{r} = t^3\bar{i} - 6t\bar{j} + 6\bar{k}$$

Find r with respect to the axis system xyz which has $\bar{i}, \bar{j}, \bar{k}$ as its unit vectors. Is $\dot{\bar{r}}$ a velocity?

2.18 If the xyz system in Problem 2.17 is rotating at $3\bar{i} + 2\bar{j} - \bar{k}$ rad/sec with respect to another system XYZ, find $\dot{\bar{r}}$ with respect to XYZ. Is $\dot{\bar{r}}$ the velocity of the point whose radius vector is $\bar{r}$ with respect to XYZ? What system is the answer of this problem referred to?

2.19 A flywheel starts from rest and accelerates counterclockwise at a constant 3 rad/sec$^2$. After six seconds the point P on the rim of the wheel has reached the position shown in the sketch. What is the velocity of point P with respect to the fixed XYZ system shown?




2.20 If

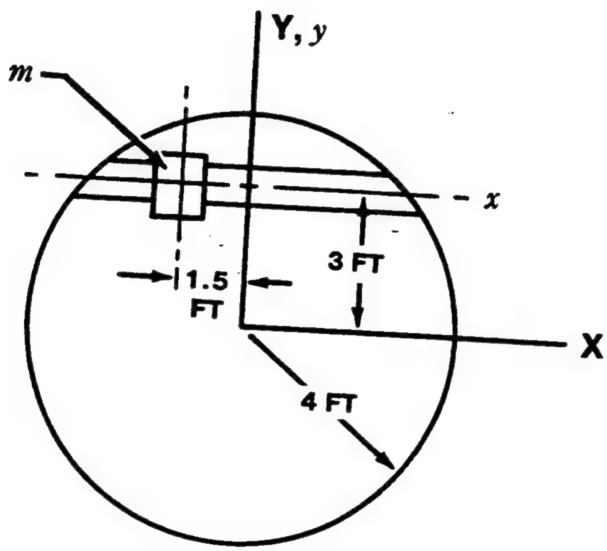
$$\bar{A} = 3t^2\bar{i} - t\bar{j}$$
$$\bar{B} = -6t\bar{i} + t\bar{k}$$

Find $d(\bar{A}\cdot\bar{B})/dt$ relative to the system having $\bar{i}$, $\bar{j}$, and $\bar{k}$ as its unit vectors. Is the answer a vector?

2.21 A small body of mass m slides on a rod which is a chord of a circular wheel as shown below. The wheel rotates about its center with a clockwise velocity 4 rad/sec and a clockwise angular acceleration of 5 rad/sec$^2$. The body m has a constant velocity on the rod of 6 ft/sec to the right. Relative to the fixed axis system XY shown below, find the absolute velocity and acceleration of m when at the position shown. Hint: Let xy system rotate with the disk as shown.

$\bar{\omega} = -4\bar{k}$

$\bar{\omega} = -5\bar{k}$



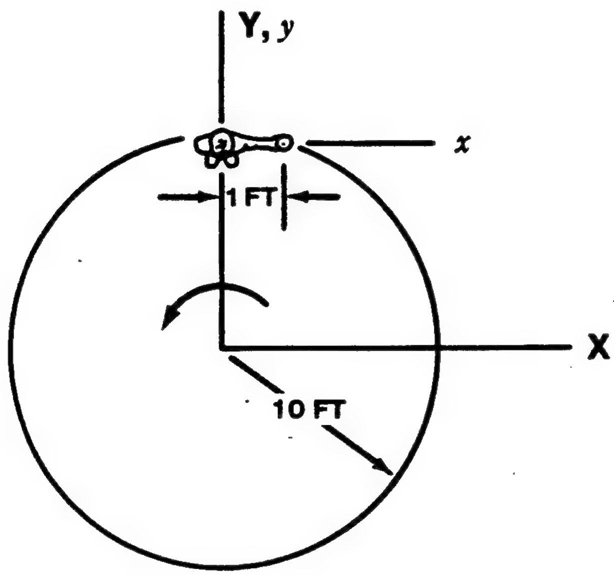
2.22 A small boy holding an ice cream cone in his left hand is standing on the edge of a carousel. The carousel is rotating at 1 rad/sec counterclockwise. As the boy starts walking toward the center of the wheel, what is the velocity and acceleration vector of the ice cream cone relative to the ground XY?

Hint: Let xy be attached to the edge of the carousel.

Boy's velocity = 2 ft/sec toward center

Boy's acceleration = 1 $ft/sec^2$ toward center

Carousel's acceleration = 1 $rad/sec^2$ counterclockwise.




2.23 Solve the following equations for $x_1$, $x_2$, and $x_3$ by use of the inverse matrix.

$$x_1 + x_3 = 2$$

$$2x_2 + 2x_3 = 1$$

$$-x_1 + x_2 + 2x_3 = 3$$

2.24 For a – i let

$$[A] = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \qquad [D] = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$[X] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad [Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Compute

a. [C] – [A]
b. [A] [B]
c. ([A] [B]) [Y]
d. [A] ([B] [Y])
e. [A] [C]
f. [C] [A]
g. $[X]^T$ [B]
h. $[X]^T$ ([A] [X])
i. [X] $[D]^T$

2.25 If $[A] = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, Find [A][A]

2.26 Find x, y, and z

$$\begin{aligned} -4x - 3y - 3z &= 1 \\ x + z &= 1 \\ 4x + 4y + 3z &= 1 \end{aligned}$$

2.27 Read the question and circle the correct answer, True (T) or False (F):

T F A vector is a quantity whose direction and sense are fixed, but whose magnitude is unspecified.

T F A scalar is a quantity with magnitude only.

T F The magnitude of a unit vector is one.

T F Zero vectors have any direction necessary.

T F A free vector can be moved along its line of action, but not parallel to itself.

T F Free vectors may be rotated without change.

T F A 3 x 2 matrix can pre-multiply a 2 x 4 matrix and the result will be a 3 x 4 matrix.

T F A 3 x 2 matrix can post-multiply a 2 x 4 matrix and the result will be a 3 x 4 matrix.

T F Multiplying a matrix by a scalar is the same as multiplying its determinant by the same scalar.

T F Identity matrices are always square.

T F $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

T F Both matrices in the preceding question are identity matrices.

T F Singular matrices can be inverted.

T F The determinant of a non-singular matrix is zero.

T F Inverting a matrix is a straightforward process.

T F $$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

T F The determinant of any matrix can be calculated.

T F The value of a determinant depends upon which row or column it was expanded about.

T F Velocity is the time rate of change of a velocity vector.

T F Acceleration is the time rate of change of a velocity vector.

T F Acceleration has to be expressed in (referred to) unit vectors of an inertial reference system.

T F Bodies moving with pure translation only do not rotate.

T F Reference systems are considered to be non-deformable rigid bodies.

T F [A] [B] = [B] [A], if the two matrices are conformable for multiplication on the left hand side of the equation.

T F $|\bar{v}| = |-\bar{v}|$

T F $|\hat{a}| = |\bar{i}|$

T F The magnitude of $\bar{A}/|\bar{A}|$ is equal to $\bar{B}/|\bar{B}|$

T F $2(3\bar{A}) = 5\bar{A}$

T F $\bar{i}$, $\bar{j}$, and $\bar{k}$ are orthogonal.

T F $|\overline{PQ}|$ is the distance between points P and Q.

T F $\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}$

T F If $\bar{A} \cdot \bar{B}$ is zero and neither $\bar{A}$ nor $\bar{B}$ are zero, then $\bar{A}$ and $\bar{B}$ must be parallel.

T F $\bar{i} \cdot \bar{i} = 1$

T F $\bar{A} \times \bar{B} = \bar{B} \times \bar{A}$

T F $\bar{A} \times \bar{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$

2.28 Define:

Determinant

Vector

Scalar

Free vector

Bound vector

Velocity vector of a particle

Unit vector

Zero vector

Parallel vectors

Position vector

Matrix

Square matrix

Column vector

Row vector

Matrix equality

Matrix conformability

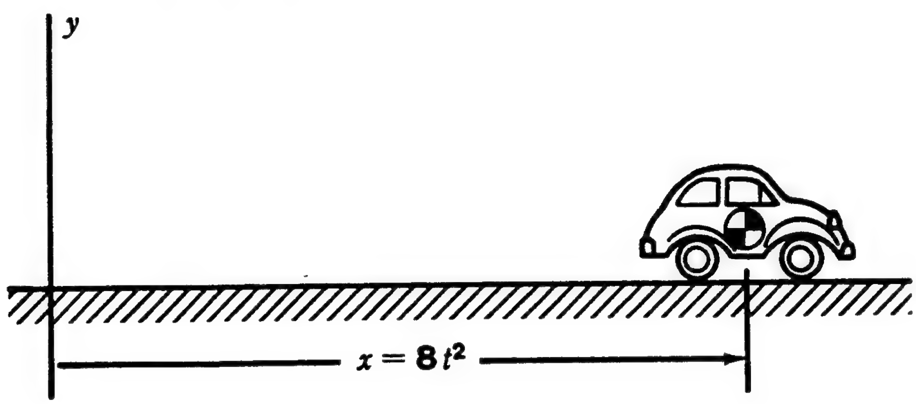
Matrix non-commutativity

Identity matrix

Transposed matrix

Singular matrix

2.29 Find the VW's velocity and acceleration vectors:



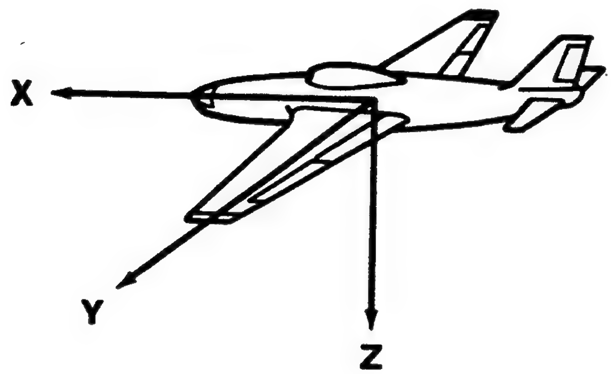

2.30 Find a unit vector parallel to

$$\bar{A} = 2\bar{i} - 3\bar{j} + 6\bar{k}$$

2.31 What is the magnitude of the following vector?

$$\bar{A} = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

2.32 Is the axis shown a "right-handed" axis system?



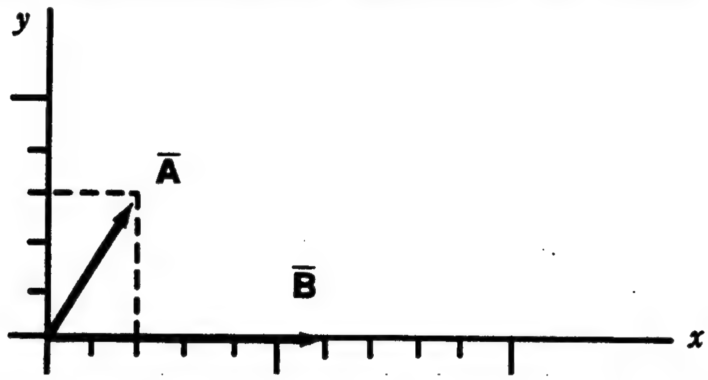

2.33 Given the following position vector, find the acceleration at time t = 0.

$$\bar{r} = 6t^3\bar{i} - 3t\bar{j} + 6t^2\bar{k}$$

2.34 Add the following vectors

$$\bar{A} = 3\bar{i}$$
$$\bar{B} = 4\bar{k}$$
$$\bar{C} = 1/2\bar{j}$$

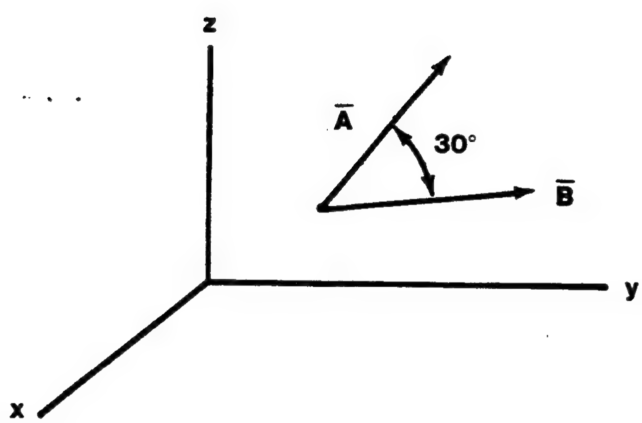
2.35 Find $\bar{A} + \bar{B}$ and the angle it makes with the x axis.




2.36 What is the angle between the two vectors given below?

$$\bar{A} = 2\bar{i} - 7\bar{k}$$
$$\bar{B} = 5\bar{i} + 2\bar{j} - 6\bar{k}$$

2.37




If $|\bar{A}| = 7$,
$|\bar{B}| = 8$,
and both vectors lie in the y - z plane, find
$\bar{A} \times \bar{B}$

2.38 Given

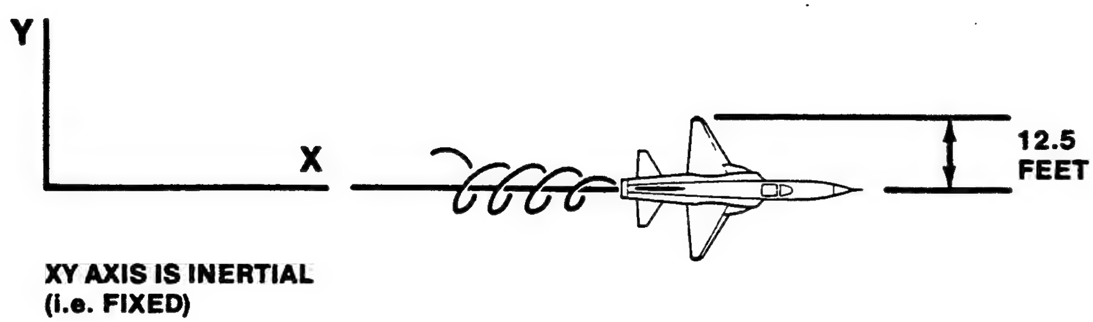
$$\bar{A} = 6\bar{i} - 2\bar{j} + \bar{k}$$
$$\bar{B} = \bar{i} + \bar{k}$$

Find

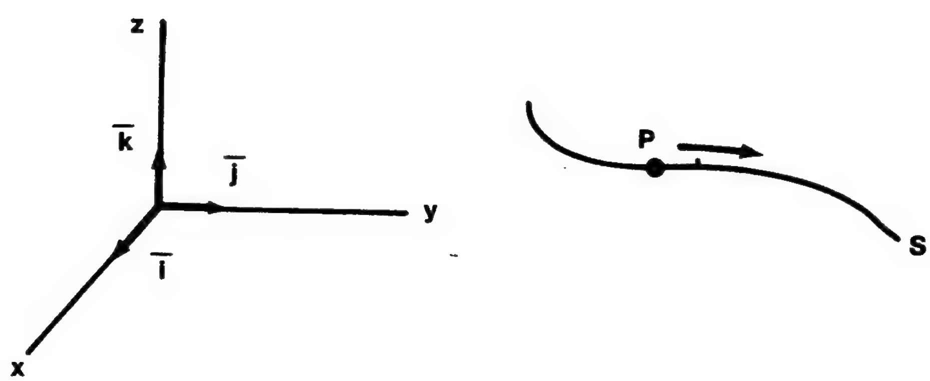
$$\bar{A} \times \bar{B}$$

2.39 The angular velocity of a rotating rigid body about an axis of rotation is given by $\bar{\omega} = 4\bar{i} + 2\bar{j} + \bar{k}$. Find the linear velocity of the Point P on the body whose position vector relative to a point on the axis of rotation is $\bar{i} - 2\bar{j} + 2\bar{k}$.

2.40 The T-38 shown is in a right continuous roll at 2 rad/sec while traveling at 480 ft/sec. Find the velocity of the wingtip light with respect to the axis shown.




2.41 The particle, P, is following a path described by : $x = 6t^2$, $y = t + 1$, $z = t^3$. Find the velocity and acceleration of P, with respect to the axis shown.



2.42 If $\bar{A} = 3\bar{i} - \bar{j} + 2\bar{k}$ and $\bar{B} = -\bar{i} + 3\bar{j} - \bar{k}$, find

a. $|\bar{A}|$

b. $|\bar{B}|$

c. $\bar{A} + \bar{B}$

d. $|\bar{A} + \bar{B}|$

e. $\bar{A} \cdot \bar{B}$

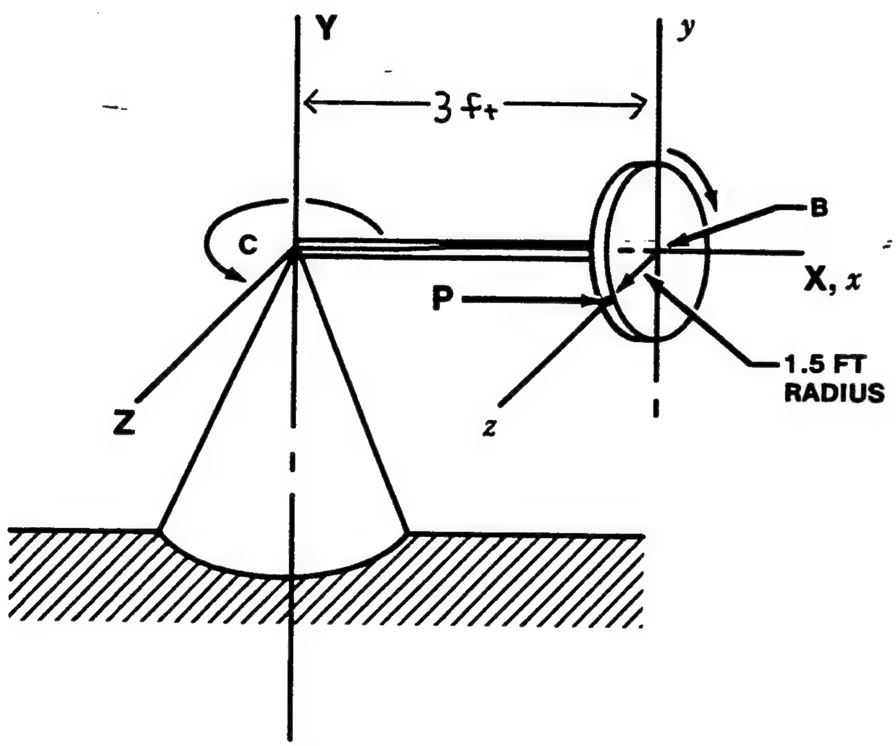
f. $\bar{A} \times \bar{B}$

g. Unit vector, $\hat{a}$, parallel to $\bar{A}$

h. $\hat{a} \cdot \bar{A}$

i. $\hat{a} \cdot \bar{B}$

2.43 The shaft is rotating counterclockwise around the cone in the XZ plane at 5 rad/sec and accelerating at 3 rad/sec$^2$. The wheel is rotating as shown at 200 rad/sec and decelerating at 50 rad/sec$^2$. Find the velocity of point P with respect to reference system C at the instant shown. Hint: Let x be fixed in the shaft, and xz and XZ planes remain coplanar.



2.44 If $[A] = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}$

Find: [A] [B] and [B] [A]

2.45 If $[A] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $[B] = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find [A] [B] and [B] [A]

2.46 $\begin{bmatrix} 3 & 6 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} =$

2.47 $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} + (3)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} =$

2.48 If $\begin{bmatrix} x & 1 \\ 2 & y + z \end{bmatrix} = \begin{bmatrix} 4 & y - z \\ 2 & 5 \end{bmatrix}$

Find x, y, and z.

2.49 If $\begin{bmatrix} 2 \\ -a \end{bmatrix} = k \begin{bmatrix} -4 \\ 2a \end{bmatrix}$

Find k.

2.50 Compute the inverse of

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

2.51 Compute the inverse of

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ 6 & 4 & 2 \end{bmatrix}$$

2.52 Compute the inverse of

$$\begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

2.53 For what value of y is this matrix singular?

$$\begin{bmatrix} 1 & 3 & y \\ 2 & 0 & -1 \\ 1 & 1 & -y \end{bmatrix}$$

2.54 Find the determinant of

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 8 & x & 0 & 0 & 0 & 0 \\ 12 & 10 & 3 & 0 & 0 & 0 \\ 1 & -1 & 6 & x^{-1} & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

2.55 If $[A] = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

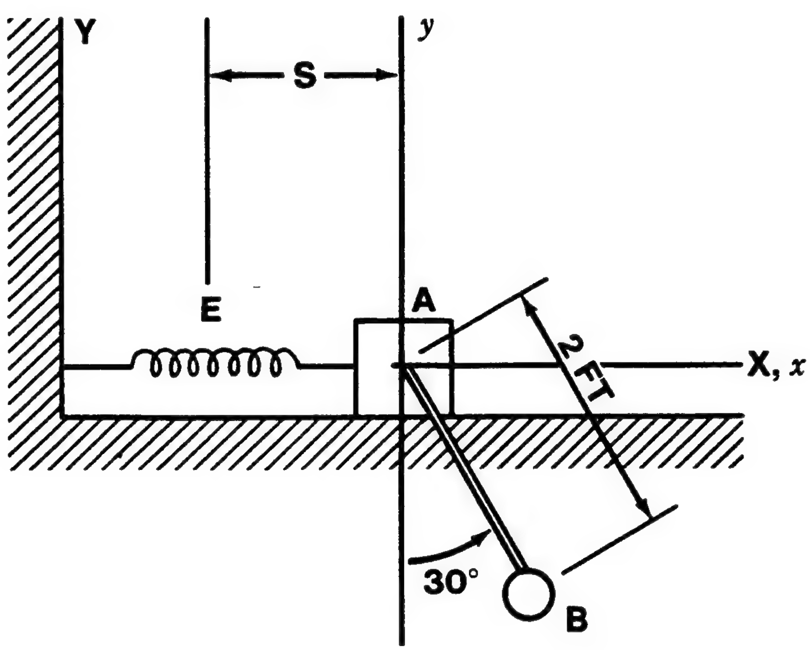
Find $[A][A]$

2.56 If

$$x + 2y + 3z = a_1$$
$$4x + 5y + 6z = a_2$$
$$7x + 8y + 9z = a_3$$

Find x, y, and z for any value of $a_1$, $a_2$, and $a_3$.
Find x, y, and z, when $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$.

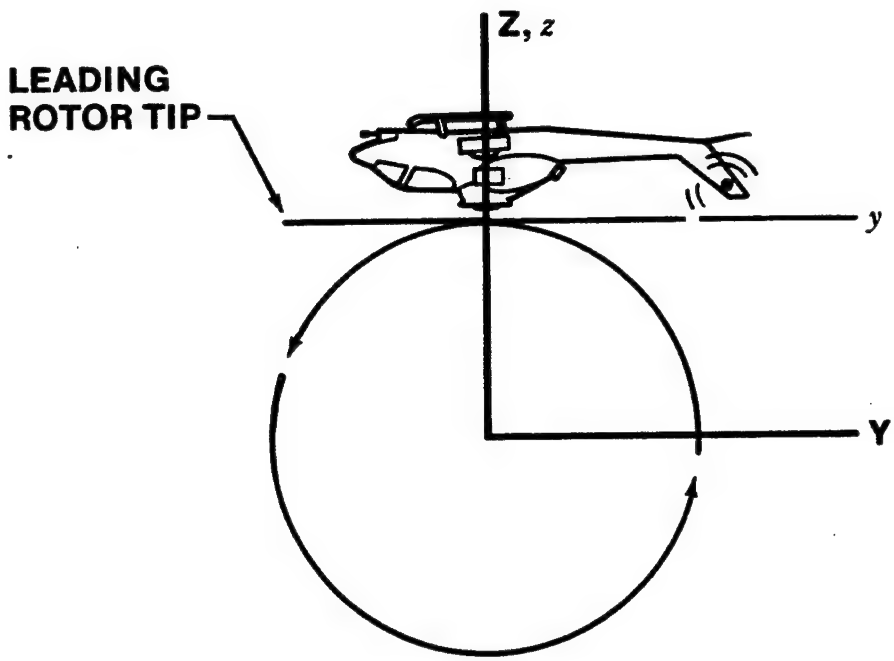
2.57 The mechanism shown below vibrates about its equilibrium position, E. At the instant shown block A has a velocity of 5 ft/sec to the right and is decelerating at 4 ft/sec$^2$ to the left. The bob B in its counterclockwise motion maintains a constant angular velocity $|\bar{\Omega}|$ of 5 rad/sec. Calculate the velocity and acceleration of the bob relative to the given XY system at the instant shown. Hint: Let the xy axis be fixed to the block A.



2.58 Capt. Marvel, US Army, is performing a loop with an angular velocity $|\bar{\omega}|$ of 1 rad/sec in his Huey Cobra to roll in on a target. At the top of the loop, the leading rotor blade is just parallel with the helicopter's centerline. The rotation of the rotor $|\bar{\Omega}|$ is 3 rad/sec counterclockwise as viewed from the top of the helicopter. At this instant, what is the velocity of the leading rotor blade tip? If Capt. Marvel were to raise the collective and accelerate the rotor speed by 3 rad/sec$^2$, this would accelerate an angular velocity of his loop by 1 rad/sec$^2$. What would the acceleration of the leading rotor blade tip be? Hint: Let the xyz system be attached to the helicopter rotor path plane as shown.

radius of loop = 1,000 ft

radius of rotor path plane = 10 ft




Hint: Let the YZ and yz planes remain coplanar.

## ANSWERS

2.1 Yes, magnitude of $\bar{V} = 1$

2.2 $\dfrac{2\bar{i} + 3\bar{j} - \bar{k}}{(14)^{1/2}}$

2.3 No, $\bar{B} = 2\bar{A}$

2.4 $\bar{F}_y = 2\bar{i} - \bar{j}$

$|\bar{F}_T| = \sqrt{5}$

2.5 $11\bar{i} - 8\bar{k}$; $\sqrt{93}$ ; $\sqrt{398}$

2.6 $\overline{PQ} = 2\bar{i} - 6\bar{j} + 3\bar{k}$

$|\overline{PQ}| = \sqrt{49}$

2.7 $\bar{A} \cdot \bar{B} = 2$

2.8 $\bar{A} \cdot \bar{B} = 28$; $\phi = 0$

2.9 $-3$; undefined $\bar{i} \cdot \bar{j} = 0$

2.10 $\bar{A} \times \bar{B} = 19\bar{i} + 17\bar{j} + 10\bar{k}$

2.11 $a = 3$

2.12 $\hat{u} = \frac{3}{7}\bar{i} - \frac{2}{7}\bar{j} + \frac{6}{7}\bar{k}$

2.13 $\bar{A} \times \bar{B} = 10\bar{i} + 3\bar{j} + 11\bar{k}$; $\bar{B} \times \bar{A} = -10\bar{i} - 3\bar{j} - 11\bar{k}$;

$(\bar{A} + \bar{B}) \times (\bar{A} - \bar{B}) = -20\bar{i} - 6\bar{j} - 22\bar{k}$

2.14 $-8\bar{i} - 6\bar{k}$; $+2\bar{i} - \bar{j}$

2.15 $\bar{\omega} = \frac{1}{10}\bar{j}$

2.16 $\overline{\omega} = .07\overline{j} - .07\overline{k}$

2.17 $\dot{\overline{r}} = 3t^2\overline{i} - 6\overline{j}$

2.18 $(\dot{\overline{r}})_{xyz} = (3t^2 - 6t + 12)\overline{i}$
$-(t^3 + 24)\overline{j}$
$-(2t^3 + 18t)\overline{k}$
in XYZ system

2.19 $\overline{V}_{xyz} = -108\overline{j}$ ft/sec

2.20 $\dfrac{d(\overline{A} \cdot \overline{B})}{dt} = -54t^2$

2.21 $\overline{V}_{P/C} = 18\overline{i} + 6\overline{j}$; $\overline{A}_{P/C} = 39\overline{i} - 88.5\overline{j}$

2.22 $\overline{V}_{P/C} = -10\overline{i} - \overline{j}$; $\overline{A}_{P/C} = -10\overline{j} - 7\overline{i}$

2.23 $x_1 = -1/4$; $x_2 = -7/4$; $x_3 = 9/4$

2.25 $$\begin{bmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{bmatrix}$$

2.26 $x = -10$; $y = 2$; $z = 11$

2.29 $\overline{V} = 16t\overline{i}$; $\overline{a} = 16\overline{i}$

2.30 $\hat{a} = \frac{2}{7}\overline{i} - \frac{3}{7}\overline{j} + \frac{6}{7}\overline{k}$

2.31 $|\overline{A}| = 7$

2.33 $\overline{a} = 12\overline{k}$

2.34 $\bar{A} + \bar{B} + \bar{C} = 3\bar{i} + 4\bar{k} + 1/2\bar{j}$

2.35 $\bar{A} + \bar{B} = 8\bar{i} + 3\bar{j}$; $\phi = 20°$

2.36 $\phi = 27.6°$

2.37 $\bar{A} \times \bar{B} = -28\bar{i}$

2.38 $\bar{A} \times \bar{B} = -2\bar{i} - 5\bar{j} + 2\bar{k}$

2.39 $\bar{V}_P = 6\bar{i} - 7\bar{j} - 10\bar{k}$

2.40 $\bar{V} = 480\bar{i} + 25\bar{k}$

2.41 $\bar{V} = 12t\bar{i} + \bar{j} + 3t^2\bar{k}$; $\bar{a} = 12\bar{i} + 6t\bar{k}$

2.42
a $\sqrt{14}$
b $\sqrt{11}$
c $2\bar{i} + 2\bar{j} + \bar{k}$
d 3
e $-8$
f $-5\bar{i} + \bar{j} + 8\bar{k}$
g $(3/\sqrt{14})\bar{i} - (1/\sqrt{14})\bar{j} + (2/\sqrt{14})\bar{k}$
h $\sqrt{14}$
i $-8/\sqrt{14}$

2.43 $\bar{V}_{P/C} = 300\bar{j} + 7.5\bar{i} - 15\bar{k}$
$\bar{A}_{P/C} = -70.5\bar{i} - 75\bar{j} - 60046.5\bar{k}$

2.44
$$[A]\,[B] = \begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix} \qquad [B]\,[A] = \begin{bmatrix} -1 & -3 & 3 \\ 6 & 3 & 0 \\ 4 & -3 & 6 \end{bmatrix}$$

2.45
$$[A]\,[B] = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$
[B] [A] Cannot do

2.46
$$\begin{bmatrix} 2 & 9 \\ 2 & 2 \end{bmatrix}$$

2.47 $\begin{bmatrix} 4 & 3 \\ 5 & 5 \end{bmatrix}$

2.48 $x = 4$; $y = 3$; $z = 2$

2.49 $k = -1/2$

2.50 $[A]^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

2.51 No Inverse

2.52 $[A]^{-1} = \frac{1}{7} \begin{bmatrix} -2 & -3 & 8 \\ 3 & 1 & -5 \\ 1 & 2 & 4 \end{bmatrix}$

2.53 $y = 1/4$

2.54 72

2.55 $\begin{bmatrix} 1 & 9 & 11 \\ 0 & -2 & 1 \\ 2 & 5 & 4 \end{bmatrix}$

2.56 No Solution

2.57 $\overline{V}_{P/C} = 13.66\overline{i} + 5\overline{j}$; $\overline{A}_{P/C} = -29\overline{i} + 43.3\overline{j}$

2.58 $\overline{V}_{P/C} = -30\overline{i} - 1000\overline{j} - 10\overline{k}$; $\overline{A}_{P/C} = -30\overline{i} - 900\overline{j} - 1010\overline{k}$